A GENERALIZED SINGULAR VALUE APPROACH TO THE LOCALIZATION OF CLOSED LOOP EIGENVALUES FOR MULTIVARIABLE DISCRETE-TIME SYSTEMS

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ABSTRACT

This paper presents a method of eigenvalue localization for discrete time systems which does not require explicit placement of the closed loop eigenvalues. The method is based on a simultaneous diagonalization accomplished via the generalized singular value decomposition (GSVD).

The GSVD is introduced and applied to the system matrix and modified control distribution matrix to show that a proper choice of the full state feedback gain matrix leads to a diagonalized closed loop system matrix. Furthermore, a particular choice of the feedback gain matrix exists which reduces the closed loop singular values as compared to those of the open loop. This fact is used to suggest an iterative algorithm for localization of the closed loop eigenvalues.

1.0 Introduction

Control engineers have long been accustomed to resorting to design methods which directly achieve, not the desired closed loop characteristics, but other closed loop characteristics which may or may not insure satisfaction of the actual specifications. The most common example of this situation is the use of modified linear Quadratic Gaussian (LQG) techniques to achieve closed loop damping and the use of LQG with loop transfer recovery (LTR) to recover robustness. While to some extent the recent development of $H^\infty$ design techniques has alleviated the problem of being forced to design from the "back door", these techniques are not completely accessible to the practicing engineer.

This paper presents a method of designing full state feedback which directly provides a bound on the damping factors of a discrete time closed loop system. The method is based on using the generalized singular value decomposition (GSVD) to obtain the simultaneous diagonalization of the system matrix and the control distribution matrix. A proper choice of the feedback gain matrix then leads to a diagonalized closed loop system matrix. The diagonal parameters are very simple functions of the diagonal elements of the feedback gain matrix.

The first topic of this paper is a discussion and derivation of the CS decomposition, from which the GSVD follows immediately. Next, the GSVD is applied to the state space representation of a discrete time system to obtain a simultaneous diagonalization of the system and control distribution matrices. A choice of the structure of the feedback gain matrix is then presented which also leads to a diagonalized closed loop system matrix.

A particular choice of the feedback gain matrix can be made which guarantees a reduction in the singular value bound on the eigenvalues over that of the open loop. This fact can be used to realize an iterative algorithm which achieves a specified bound.

2.0 The CS Decomposition

The generalized singular value decomposition is based on the CS decomposition, which has applications to signal processing [1]. Suppose $Q \in R^{mp \times np}$ is unitary and is partitioned so that

\[
Q = \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}
\]  

(1)

where $Q_{11} \in R^{m_1 \times n_1}$ and $Q_{22} \in R^{m_2 \times n_2}$. Since $Q$ is unitary,

\[
Q_{11}Q_{11}^T + Q_{12}Q_{12}^T = I, 
\]

(2)

\[
Q_{11}Q_{21}^T + Q_{12}Q_{22}^T = 0, 
\]

(3)
Q_{21}Q_{11}^T + Q_{22}Q_{12}^T = 0, \quad (4)

and

Q_{21}Q_{21}^T + Q_{22}Q_{22}^T = I. \quad (5)

Also,

Q_{11}Q_{11}^T + Q_{21}Q_{21}^T = I, \quad (6)

Q_{11}Q_{12}^T + Q_{21}Q_{22}^T = 0, \quad (7)

Q_{12}Q_{11}^T + Q_{22}Q_{21}^T = 0, \quad (8)

and

Q_{12}Q_{12}^T + Q_{22}Q_{22}^T = I. \quad (9)

Q_{11} and Q_{22} have singular value decompositions such that

Q_{11} = U_1\Sigma_{11}V_1^T, \quad (10)

and

Q_{22} = U_2\Sigma_{22}V_2^T. \quad (11)

Then from Equation (5)

Q_{21}Q_{21}^T = U_2[I - \Sigma_{22}^2]U_2^T \quad (12)

and from Equation (6)

Q_{21}Q_{21}^T = V_1[I - \Sigma_{11}^2]V_1^T. \quad (13)

It is useful to assume that p < n so that rank(Q_{22}) \leq p and since the decomposition is used for a special purpose, it is assumed that rank(Q_{11}) = p. Then from Equation (13) and the fact that the singular values of Q_{11} are in decreasing order of magnitude it is clear that the first n-p elements of \Sigma_{11} are unity so that

\Sigma_{11} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \quad (14)

where I \in R^{(n-p)\times(n-p)}, C \in R^{p\times p}, and C is diagonal. Now since the non-zero eigenvalues displayed in Equations (12) and (13) must be the same,

\Sigma_{22} = C. \quad (15)

Also, since Equations (12) and (13) define a singular value decomposition of Q_{21},

Q_{21} = \pm U_2[0 S]V_1^T, \quad (16)

where

C^2 + S^2 = I. \quad (17)

Similarly,

Q_{12} = \pm U_1[0 S]V_2^T. \quad (18)

The singular value decompositions of the elements of Q yield the following decomposition of Q itself:

Q = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad (19)

where

T_{11} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}, \quad (20)

T_{12} = \pm \begin{bmatrix} 0 & S \\ 0 & I \end{bmatrix} \quad (21)

T_{21} = \pm \begin{bmatrix} 0 & S \end{bmatrix} \quad (22)

T_{22} = C. \quad (23)

Now, since the left hand side of Equation (19) is the product of unitary matrices, the right hand side must be unitary. This requirement means that either

T_{12} = \begin{bmatrix} 0 & -S \end{bmatrix} \quad (24)

T_{21} = \begin{bmatrix} 0 & S \end{bmatrix} \quad (25)

or

T_{12} = \begin{bmatrix} 0 & S \end{bmatrix} \quad (26)

T_{21} = \begin{bmatrix} 0 & -S \end{bmatrix}. \quad (27)

For convenience sake only, it is assumed that Equations (26) and (27) hold.
The decomposition of Equation (19) is called the CS (cosine-sine) decomposition because Equation (17) implies that the elements of C are cosines and the elements of S are sines. The CS decomposition yields the GSVD immediately, as is seen in the next section.

3.0 The GSVD and Simultaneous Diagonalization

The GSVD is a simple application of the CS decomposition to the unitary factor of the QR decomposition of a matrix. In particular,

\[
\begin{bmatrix}
A^T \\
B^T
\end{bmatrix} = QR
\]

(28)

where \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}, Q \in \mathbb{R}^{(m \times n) \times (m \times n)}, \) and \( R \) is upper triangular. \( R \) is assumed to have rank \( n \) and the principal \( n \times n \) submatrix of \( R \) is assumed to be nonsingular. Denoting this submatrix by \( R_n \), then \( A \) and \( B \) can be written as

\[
A^T = R_1^T u_1 v_1^T
\]

(29)

\[
A^T B = R_2^T v_2 T_{12}^T u_2^T
\]

(30)

where \( u_1, v_1, v_2, T_{11}, \) and \( T_{12} \) are elements of the CS decomposition of \( Q \). Equations (29) and (30) are a type of simultaneous diagonalization of both matrices. The next step in the eigenvalue localization procedure is to derive bounds on the singular values of \( A - BK \) when

\[
x(k+1) = Ax(k) + Bu(k).
\]

(31)

\( K \) is to be chosen so that with

\[
u(k) = -Kx(k),
\]

(32)

the eigenvalues of \( A - BK \) satisfy

\[
| \lambda_i(A - BK) | \leq e^{\sigma T}
\]

(33)

where \( \sigma \) is a desired damping factor and \( T \) is the sampling period. Equation (33) is insured by requiring that

\[
\sigma_{\text{min}}(A - BK) \leq e^{\sigma T}
\]

(34)

where \( \sigma_{\text{max}}[\cdot] \) denotes the maximum singular value.

The basis of the eigenvalue localization procedure is to obtain a bound on the maximum singular value of the closed loop system matrix. The first step is to note that \((A^T + A^T BF)^{-1}\) can be written as [2]

\[
(A^T + A^T BF)^{-1} = A - B(FB + I)^{-1}FA.
\]

(35)

The matrix \( F \) is considered to be intermediate and

\[
K = (FB + I)^{-1}FA.
\]

(36)

Equation (35) implies that

\[
1/\sigma_{\text{min}}(A^T + A^T BF) = \sigma_{\text{max}}(A - BK)
\]

(37)

so that decreasing the maximum singular value of \( A - BK \) can be accomplished by increasing the minimum singular value of the left hand side of Equation (37).

Equations (29) and (30) can be used to write

\[
A^T + A^T BF = R_1^T v_1 (T_{11} + T_{12}^T T_{12}) u_1^T
\]

(38)

when \( F \) is chosen so that

\[
F = U_2^T S U_1^T
\]

(39)

and

\[
T_{12}^T = [0 \; \Sigma].
\]

(40)

The motivation for Equation (40) is that if \( \Sigma \) is diagonal, then so is \( T_{12}^T = T_{12}^T T_{12} \). The singular values are then simply the absolute values of the diagonal elements, which are in turn linear functions of the elements of \( \Sigma \). Also,

\[
\sigma_{\text{min}}(A^T + A^T BF) > \sigma_{\text{min}}(R_1^T) \sigma_{\text{min}}(T_{11} - T_{12}^T T_{12}).
\]

(41)

Some manipulation of Equation (41) also implies that

\[
\sigma_{\text{max}}(A - BK) \leq \sigma_{\text{max}}(A) / \sigma_{\text{min}}(T_{11} - T_{12}^T T_{12}).
\]

(42)
Unfortunately, the structure of $T_1$ and $T_2$ do not allow for complete specification of the minimum singular values on the right hand side of Equation (42). However, $T_1$ and hence $T_2$ can be chosen so that
\[
\sigma_{\text{min}}(A - BK) \leq \sigma_{\text{max}}(A)
\]  
so that the closed loop singular values are bounded by the maximum open loop singular value. Equality in Equation (43) appears to occur only for degenerate cases. The precise conditions for equality are a subject of future work. The situation for which strict inequality prevails is the basis for an iterative algorithm suggested in the next section.

At this point, it should be noted that the choice of $T_2$ to be diagonal is quite restrictive. It does, however, provide directly the singular values of $F$, which can be used to derive a norm bound for $K$. The precise form of this bound is also a topic of future investigation.

4.0 An Iterative Algorithm for Eigenvalue Localization

The bound of Equation (43) forms the basis of an iterative algorithm for achieving a prespecified damping factor $\sigma$. This algorithm is described by the following steps:

1. Set $K_0 = 0$
2. Calculate the GSVD of $A^T$ and $B^T$ to obtain $T_1$, $T_2$, $U_1$, and $U_2$. 
3. Solve $T_1 + T_2^T T_2 = I$ to find $T_2$ such that Equation (43) holds.
4. Set $F = U_2 T_2 U_1^T$ and calculate $K$ from Equation (36).
5. Set $K_i = K_i + K$
6. Let $A_i = A - BK_i$
7. Calculate the singular values of $A_i$
8. If $\sigma_{\text{max}}(A_i) > \sigma_0$ then set $A = A_i$ and repeat steps 2-8
9. If $\sigma_{\text{max}}(A_i) \leq \sigma_0$ then the desired feedback gain matrix is $K_i$.

5.0 Conclusions

The generalized singular value decomposition has been used to obtain a feedback gain matrix which guarantees a reduction of the maximum singular value of the closed loop system matrix below that of the open loop system matrix. This reduction can be used in iterative procedures for calculating feedback gain matrices which yield closed loop system matrices whose maximum singular values satisfy a specified upper bound. This upper bound in turn can be interpreted as a specification on the closed loop damping factors of all closed loop eigenvalues. There is no need to specify the precise locations of the closed loop eigenvalues. An algorithm for calculating feedback gain matrices to achieve a specified damping factor has been suggested.

Several questions are open regarding the procedure outlined in this paper. First, it may be possible to implement a non-iterative procedure by relaxing slightly the structural constraints on the intermediate matrix $F$. Secondly, the precise conditions in which the bound of Equation (43) is an equality are unknown. These conditions will be investigated to determine causes of possible premature termination of the suggested iterative algorithm.

6.0 References