Errors in frequency-domain processing of images

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INTRODUCTION

Practical techniques for the determination of image spectra have been developed and become popular in the past few years. Both optical processing systems and digital computers can be used to perform linear filtering via the frequency domain. Optical processing systems use Fourier-transforming lenses and coherent light. Digital computer software uses the Cooley-Tukey algorithm to advantage, while computer hardware must be augmented by optical scanning devices that interface with images. Processing errors arise in both types of systems, but for different reasons. In this paper we present some results concerning errors in the spatial frequency domain.

Two-dimensional Fourier analysis

To facilitate later discussions, we shall review briefly the key relations in two-dimensional Fourier analysis. The Fourier transform of \( f(x, y) \) is defined as

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i (ux + vy)} \, dx \, dy .
\]

(1)

The inversion relation is then given by

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi i (ux + vy)} \, du \, dv .
\]

(2)

If \( f(x, y) \) is nonzero only inside a finite rectangular area \( 0 \leq x \leq T_x, 0 \leq y \leq T_y \), then a two-dimensional Fourier series may be used to represent \( f(x, y) \) in that area. In particular,

\[
f(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{2\pi i (mx/T_x + ny/T_y)} .
\]

(3)

where

\[
a_{m,n} = \frac{1}{T_x T_y} \int_{0}^{T_x} \int_{0}^{T_y} f(x, y) e^{-2\pi i (mx/T_x + ny/T_y)} \, dx \, dy .
\]

(4)

When \( f(x, y) \) is impulsive and of the form

\[
f(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \delta(x - mT_x, y - nT_y),
\]

(5)

where \( \delta \) is the Dirac delta function and \( T_x = M \tau_x \) and \( T_y = N \tau_y \), then a discrete Fourier transform is appropriate. The discrete Fourier transform of the discrete function \( \beta_{m,n} \) is given as

\[
\lambda_k,i = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \beta_{m,n} e^{-2\pi i (km/M + in/N)},
\]

(6)

where \( k = 0, 1, \ldots, M - 1 \) and \( i = 0, 1, \ldots, N - 1 \). The inversion relation is

\[
\beta_{m,n} = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{i=0}^{N-1} \lambda_k,i e^{2\pi i (km/M + in/N)},
\]

(7)

Fourier transformation

Optical processing systems

When a film transparency of complex amplitude \( f(x, y) \) is illuminated with collimated monochromatic light at the front focal plane of a double-convex lens, the light amplitude at the back focal plane will be
\[ F(x/\lambda d, y/\lambda d) = F(u, v), \] the Fourier transform of \( f(x, y) \). In this relation, \( x \) and \( y \) are spatial coordinates, \( \lambda \) is the wavelength of light, \( d \) is the focal length of the lens, and \( u \) and \( v \) are spatial frequencies.

**Digital processing systems**

The Cooley-Tukey algorithm reduces the number of basic operations in the calculation of discrete Fourier transforms from \( N^2 \) to \( 2N \log_2 N \), where \( N \) is the number of sample points involved, and a basic operation is defined to be a complex multiplication followed by a complex addition. This time-saving reduction has made the calculation of image spectra practical for digital machines.

A device, which can act as an interface between images on film and the digital computer, is needed as auxiliary equipment. Precision flying-spot scanners, such as the one built by Professor W. F. Schreiber at M.I.T., are ideal for this purpose.

**Linear filtering**

**Optical processing systems**

The simplest optical processing system capable of doing linear filtering uses two double-convex lenses and two film transparencies aligned along a path of collimated coherent light. An input film is placed at the front focal plane of the first lens, while the filter-function transparency is placed at the back focal plane of the first lens. The front focal plane of the second lens is coincident with the back focal plane of the first lens. When no errors are present, the light amplitude at the back focal plane of the second lens, except for a possible change of scale, is

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) H(u, v) e^{i2\pi (ux+vy)} \, du \, dv ,
\]  

(8)

where \( H(u, v) \) is the filter function, and \( F(u, v) \) is the Fourier transform of the input image.

Complex frequency-domain filters for optical filtering may be made by varying the density of film according to a desired magnitude function and varying the film thickness to regulate phase. Practical difficulties in varying film thickness in this direct method have led to the methods of Vander Lugt, Lohmann, and Lee. These methods permit complex filtering using positive real filters. In the Lohmann and direct methods, noise in the filter transparency can be modeled approximately as independent noise that is being added to the real and imaginary parts of the filter function.

Vander Lugt used a reference beam of coherent light in recording his complex filter on film. The function recorded on film, which is non-negative, is

\[
S(u, v) = |A e^{i\alpha x} + H(u, v)|^2
\]

\[
= A^2 + |H(u, v)|^2 + AH^*(u, v) e^{i2\pi v} + AH(u, v) e^{-i2\pi v} ,
\]  

(9)

where the asterisk denotes complex conjugation. The impulse response of \( S(u, v) \) is

\[
s(x, y) = A^2 \delta(x, y) + R_h(x, y) + Ah(-x-a, -y) + Ah(x-a, y),
\]  

(10)

where \( h(x, y) \) is the impulse response of \( H(u, v) \), and \( R_h(x, y) \) is the autocorrelation function of \( h(x, y) \). Multiplication of \( F(u, v) \) by \( S(u, v) \) corresponds to the convolution of \( f(x, y) \) with \( s(x, y) \). If \( f(x, y) \) and \( h(x, y) \) are of finite spread, then the constant \( a \) can be chosen to produce the desired output \( g(x, y) \) without interference, but displaced along the \( x \)-axis in the output plane.

The filter of Lohmann has only binary transmittance values. Clear slits for light transmission are placed on film to synthesize complex transmission functions. The slit area determines the magnitude of light transmission. Varying the slit position changes the phase of the light transmitted through the slit.

Lee's method for producing complex filters on film is similar to Lohmann's method, except that it provides for a continuous variation in transmittance. Lee's filter uses four non-negative sample points placed along a line to construct a complex sample point. The positions of the four sample points result in transmission phases of 0, \( \pi/2 \), \( \pi \), and \( 3\pi/2 \), respectively. By adjusting the transmission amplitudes for the four points, any desired complex transmittance can be obtained.

**Digital processing systems**

Linear filtering is accomplished on the digital computer by Fourier transformation, followed by multiplication, followed by Fourier inversion. Round-off error occurs in this process. A crude model for the error is independent white noise added during the computation.

**Quantitative effects of frequency-domain errors**

**Additive noise**

If independent noise is added to the real and imaginary parts of \( F(u, v) \) [Equation (2)], then independent noise
of the same power will be present in the reconstructed image \( f(x, y) \) upon Fourier inversion. This is true because of Parseval's theorem. In particular, independent white noise transforms to independent white noise of the same power. Independent white noise added to the magnitude of \( F(u, v) \) also results in contamination of \( f(x, y) \) by independent white noise. The model of independent white noise can be used as a first approximation for grain noise in film, and for round-off error in digital computations.

**Multiplicative noise**

When Vander Lugt or Lee filters are used for linear filtering, \( g(x, y) \) [Equation (8)] is contaminated by a noise equal to

\[
n(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) N(u, v) e^{i2\pi (ux + vy)} \, du \, dv,
\]

where \( N(u, v) \) is the film grain noise added to \( H(u, v) \), and is assumed to be independent. Under the condition

\[
\overline{N(u, v)N^*(a, \beta)} = \sigma^2 \delta(u - a, v - \beta),
\]

where \( \sigma^2 \) is a positive constant, and the bar denotes ensemble average, it may be shown that

\[
n(x, y) n^*(r, s) = \sigma^2 R_f(x - r, y - s),
\]

where \( R_f \) is the autocorrelation function of the input image \( f(x, y) \). Film grain noise causes errors in \( |H(u, v)| \), the magnitude of \( H(u, v) \), for the direct method filter, while inaccuracy in the slit area has the same effect with the Lohmann filter. In these cases the filter becomes

\[
\tilde{H}(u, v) = |H(u, v)| + N(u, v) e^{i\phi(u, v)},
\]

where \( \phi(u, v) \) is the phase of \( H(u, v) \). If \( N(u, v) \) is independent and obeys Equation (12), then the noise in the filtered image \( \hat{g}(x, y) \) is given by

\[
n(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) N(u, v) e^{i2\pi (ux + vy)} \, du \, dv
\]

and we again have

\[
n(x, y) n^*(r, s) = \sigma^2 R_f(x - r, y - s).
\]

**Phase noise**

Spurious film thickness variations cause errors in \( \phi(u, v) \) for filters made by the direct method. Inaccuracy in the slit positions has the same effect in the Lohmann method.

When phase noise occurs, the filter function will be given by

\[
\tilde{H}(u, v) = |H(u, v)| e^{i\phi(u, v) + \phi(u, v)}.
\]

The output of the filtering system then is

\[
\hat{g}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) H(u, v) e^{i\phi(u, v)} e^{i2\pi (ux + vy)} \, du \, dv.
\]

The noise output of the system is

\[
n(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) H(u, v) [e^{i\phi(u, v)} - 1] e^{i2\pi (ux + vy)} \, du \, dv.
\]

If the noise is small \( (N \ll 1) \), independent, and satisfies Equation (12), then it follows that

\[
n(x, y) n^*(r, s) \approx \sigma^2 R_f(x - r, y - s),
\]

where \( R_f \) is the autocorrelation function of \( g(x, y) \), the ideal output image.

Under the condition that \( N(u, v) \) is Gaussian, Anderson\(^7\) has shown that

\[
[n(x, y)]^2 = 2R_f(0, 0) (1 - e^{-\sigma^2/2}),
\]

where \( \sigma^2 \) is the phase noise power measured in \((\text{rad/sec})^2\). Under the assumption that \( \phi(u, v) \) is uniformly

![Figure 1—Q as a function of phase signal-to-noise ratio](https://www.computerhistory.org)
distributed from \(-\pi\) to \(\pi\), the phase function \(\phi(u, v) = \phi(u, v) + N(u, v)\) has a signal-to-noise ratio

\[
\frac{S}{N} = \frac{\pi^2}{3\sigma^2}.
\] (22)

It follows from Equations (21) and (22) that the ratio of phase signal-to-noise ratio to image signal-to-noise ratio is

\[
Q = \frac{2\pi^2}{3\sigma^2} \left(1 - e^{-\pi^2/\sigma^2}\right).
\] (23)

This ratio is plotted in Figure 1 against phase signal-to-noise ratio.

**Quantization noise**

Images on film have limited brightness ranges, because of film characteristics. Fourier-transforming functions that represent film brightness variation normally lead to functions with much wider dynamic range. Typically, television quality images have a 30–50 dB disparity between the energy in the lowest
Spatial frequencies and the energy in the highest spatial frequencies of the spectrum. Most of the energy tends to reside in a small area of the total spectrum and is low-frequency. Linear quantization in the frequency domain without a fine quantization grain therefore causes not only a large mean-squared image error, but also results in high percentage errors in the middle and high frequencies. Since it is known that the response of the human visual system is poorest at low and high spatial frequencies and peaks at middle-range frequencies, we can assume that linear quantization in the frequency domain will yield images of poor quality. Improvement in mean-squared error and picture quality can be obtained by using nonlinear quantization with smaller quantization intervals at low amplitude levels and larger intervals at high amplitude levels.

**Subjective effects of frequency-domain errors**

**Experimental system**

The human visual system is highly nonlinear.

![Additive noise](https://www.computerhistory.org)
Although, for some purposes, quantities such as mean-squared error and resolution are useful in describing images, it is recognized that good parameters have yet to be discovered to describe image quality. With this in mind, the simulation of frequency-domain error situations was undertaken.

We recorded the test image (Figure 2) on magnetic tape, using the flying spot scanner built by Professor William F. Schreiber of the Research Laboratory of Electronics, M.I.T. Brightness was quantized to 8 bits in the process, and a sample array size of $128 \times 128$ samples was used. The noise environments were simulated on the IBM 360-65 general-purpose digital computer, and the data recorded on magnetic tape provided an input. Before display the processed outputs were extended to $256 \times 256$ arrays by using two-dimensional linear interpolation.

Figure 4—Multiplicative noise
Results

The image in Figure 3 results when independent white noise with a Gaussian probability density is added in the frequency domain. This noise addition, as can easily be shown, is statistically equivalent to adding the same noise directly to the image brightness function. Figure 4 is an image corrupted by multiplicative frequency-domain noise, which is white, independent, and Gaussian. Phase noise has altered the input image to produce the image of Figure 5. The phase noise is also white, independent, and Gaussian. What is interesting about the images of Figures 3-5 is that the signal-to-noise ratio is the same for all noise additions, 15 dB.

The image in Figure 6 is a 5-bit, linearly quantized version of the input image. To compare this image with linear quantization in the frequency domain, Figure 7 is given. The spectrum magnitude and phase of the image of Figure 7 were both linearly quantized to 5 bits.

Figure 5—Phase noise
A nonlinear quantization of the spectrum magnitude can be performed to improve the quality of this image. Choosing quantization intervals on a logarithmic scale for magnitude quantization and retaining linear quantization for phase yield the image of Figure 8, which is also a 5-bit image. The signal-to-quantization-noise ratios in Figures 7 and 8 were measured and found to be 9.78 and 13.90 dB, respectively. The images in Figures 9 and 10 are presented to illustrate the effects of magnitude and phase quantization separately. The image in Figure 9 has a spectrum magnitude that is quantized to 3 bits on a logarithmic scale, while phase has been undisturbed. In the case of the image in Figure 10, the phase has been uniformly quantized to 3 bits, while the magnitude has not been changed.

The dynamic range of an image spectrum can be partially characterized by the one-dimensional functions $|F(u, o)|$ and $|F(o, v)|$. These functions are plotted...
Figure 7—Linear spectrum quantization (3-bits)
Figure 8—Nonlinear spectrum quantization (5-bits)
Figure 9—Nonlinear magnitude quantization (3-bits)
Figure 10—Uniform phase quantization (3-bits)
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for the test image in Figure 11. Linear interpolation has been performed between data points to make the functions shown continuous.

Remarks

The theoretical predictions concerning additive, multiplicative, and phase noise are confirmed by the appearances of the images in Figures 3, 4, and 5. The image noise for the additive case (Figure 3) clearly has a white noise appearance. The appearances of multiplicative and phase noise were predicted by Equations (16) and (20). These equations stipulate that the power density spectra of image noise for the multiplicative and phase cases are identical, except for a constant, to the power density spectrum of \( F(u, v) \). The energy of \( F(u, v) \) (Figure 11) is primarily at low spatial frequencies. Therefore, the noise in the images of Figures 4 and 5 is primarily low-frequency noise.

Linear quantization of \( |F(u, v)| \) severely limits image resolution when an extremely small quantization interval is not used. Only low-frequency components are nonzero after linear frequency-domain quantization of the test image to 5-bits (Figure 7). Had Figure 11 been available before quantization, the resolution reduction could have been predicted. Figure 11 could also have been used to predict the improvement in mean-square error and image quality when logarithmic instead of linear quantization is used for \( |F(u, v)| \).

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