Degradation analysis of digitized signal transmission

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INTRODUCTION

In communication systems signals are subjected to various kinds of noises. Although extensive studies of error control methods have been conducted, there is little information available as to the effects of noise upon the recovered and processed signal.

This paper presents some results of the degradation analysis of digitized, sampled analog signals transmitted through a digital communication system shown in Figure 1. This analysis was accomplished by a theoretical investigation and a digital computer simulation of the problem.

The analog signal was a Rayleigh function. Its sampled version, was digitized and mixed with noise consisting of an independent uniformly distributed random binary sequence. At the receiver the corrupted digital signal was converted to its analog equivalent and analyzed.

Two different measures of degradation were used in the analysis. The first measure was the normalized rms difference between sample pairs; the second measure was the normalized rms difference between the spectral components associated with each pair. This second measure was selected in order that one, using a computer to perform a spectral analysis, could better relate the degradation to the channel noise.

ANALOG SIGNAL CONSIDERED FOR DEGRADATION ANALYSIS

The signal considered in this paper is a waveform of the fundamental Rayleigh mode expressed in equation (1) and shown in Figure 2,

$$f(t) = \frac{-t^2/\alpha}{\alpha} \cdot e^t$$

(1)

where $\alpha$ is the attenuation constant. It is a well-known fact that the fundamental Rayleigh mode has its energy concentrated in the range of 0.15 to 10 cps. Using Nyquist's Sampling Theorem, the rate of sampling is chosen as 20 samples per second.

Degradation analysis in sampling

Consider the analog signal $f(t)$ from equation (1). When the analog signal $f(t)$ is sampled at the rate of $n$ samples per second, the sampled data can be represented as:

$$f(t) \rightarrow X(iT)$$

(2)

where $T = \frac{1}{n}$ seconds, and where $i = 0, 1, 2, \ldots, m$. The binary sequence can be obtained by expanding $X(iT)$ into $k$ bits. Then, the rate of the digitized signal $X_i(i)$ is $nk$ bits per second. These curves are shown in Figure 3(a) and (b).

Whenever a binary sequence $X_i(i)$ is transmitted through a communication system such as that shown in Figure 1, it will be corrupted by noise. For uniformly distributed random binary noise, the communication system may be theoretically modeled after the binary symmetrical channel. The binary symmetrical channel (shown in Figure 4) is the model of the overall transmission link; the conditional probabilities of a 0 or a 1 having been sent, given that a 0 was received, are $p$ and $q$, respectively.

In this paper, the system noise $N_i(i)$ is a random binary sequence, as shown in Figure 3(c). The received binary sequence $Y_i(i)$ with rate $nk$ (bits/second) is the logical sum of $X_i(i)$ and $N_i(i)$ given in equation (3) and shown in figure 3(d).

$$Y_i(i) = X_i(i) \oplus N_i(i)$$

(3)

The signal recovered by digital-to-analog conversion $Y(iT)$ is

$$Y(iT) = \sum_{i=1}^{k} Y_i(i) \cdot 2^{(i-1)}$$

(4)

as shown in Figure 3(e). Note that the binary point is between the third and fourth leftmost bits of $Y_i(i)$. 

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Truncation errors in sampling

An error in the representation of a sampled signal \( X(iT) \) is inevitable. The amount of error \( E \) is found by taking the root mean square of the difference of the original and the recovered signals:

\[
E = \sqrt{\sum_{i=1}^{m} [T \cdot Y(iT) - T \cdot X(iT)]^2} \quad (5)
\]

The normalized error is obtained by taking the ratio of the rms value to the total area \( a \) under the original digitized curve \( X(iT) \):

\[
a = \sum_{i=1}^{m} T \cdot X(iT) \quad (6)
\]

The normalized error \( \bar{E} \) is

\[
\bar{E} = \frac{\sum_{i=1}^{m} [Y(iT) - X(iT)]^2}{\sum_{i=1}^{m} X(iT)} \quad (7)
\]

The maximum truncation error associated with each sampling point is

\[
\Delta_{\text{max}} = 2^{-k'}
\]

where \( k' \) is the number of bits to the right of the binary point of the binary represented of the sampled signal \( X(iT) \). The maximum truncation error \( E_{\text{trunc}} \) is obtained by using equation (5)

\[
E_{\text{trunc}} = T \sqrt{\sum_{i=1}^{m} [(X(iT) + \Delta_{\text{max}}) - X(iT)]^2}
\]

\[
= T \sqrt{\Delta_{\text{max}}} \quad (8)
\]

The total area \( a \) under the curve is obtained from equation (6)

\[
a = \sum_{i=1}^{m} X(iT) \cdot T \quad (9)
\]

and the normalized truncation error \( \bar{E}_{\text{trunc}} \) is obtained by using equation (7).

\[
\bar{E}_{\text{trunc}} = \frac{\sqrt{m} \Delta_{\text{max}}}{\sum_{i=1}^{m} X(iT)} = \frac{\sqrt{m} 2^{-k'}}{\sum_{i=1}^{m} X(iT)} \quad (10)
\]

The error due to transmission noise is found by taking the root mean square of the difference of the original and noise-corrupted signals. This error also includes the truncation error. The noise error and the normalized noise error are obtained by using equations (5), (6) and (7).

The maximum noise error bound can be found as follows: The maximum noise error associated with each transmitted digital sequence is

\[
\delta_{\text{max}} = 2^{(k''+1)} \cdot q
\]

where \( k'' \) is the number of bits to the left of the binary point of the transmitted sequences. The noise error \( E_{\text{noise}} \) is obtained from equation (5)

\[
E_{\text{noise}} = T \sqrt{\sum_{i=1}^{m} [(X(iT) + \Delta_{\text{max}} + \delta_{\text{max}}) - X(iT)]^2}
\]

\[
= T \sqrt{\Delta_{\text{max}} + \delta_{\text{max}}} \quad (12)
\]

and the normalized noise error, \( E_{\text{noise}} \), is obtained using equation (7)

\[
\bar{E}_{\text{noise}} = \frac{\sqrt{m} (\Delta_{\text{max}} + \delta_{\text{max}})}{\sum_{i=1}^{m} X(iT)} = \frac{\sqrt{m} [2^{-k''} + 2^{(k''+1)} \cdot q]}{\sum_{i=1}^{m} X(iT)} \quad (13)
\]

Degradation analysis of Fourier transform

The Fourier coefficients for the original signal are defined as

\[
a_n = \frac{2}{m} \sum_{i=0}^{m-1} X(iT) \cos \frac{2\pi ni}{m}; n = 0, \ldots, m-1
\]

\[
b_n = \frac{2}{m} \sum_{i=0}^{m-1} X(iT) \sin \frac{2\pi ni}{m}; n = 0, 1, \ldots, m-1
\]

The original time function \( f(t) \) is represented as

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{m-1} \left( a_n \cos \frac{2\pi ni}{m} + b_n \sin \frac{2\pi ni}{m} \right)
\]
and the power spectrum is

$$\rho_n = \sqrt{a_n^2 + b_n^2}; \quad n = 0, \ldots, \frac{m}{2} - 1$$  \hspace{1cm} (17)

The Fourier coefficients for the recovered signal are

$$a_n' = \frac{2}{m} \sum_{i=0}^{m-1} Y(iT) \cos \frac{2\pi ni}{m}; \quad n = 0, \ldots, \frac{m}{2} - 1$$  \hspace{1cm} (18)

$$b_n' = \frac{2}{m} \sum_{i=0}^{m-1} Y(iT) \sin \frac{2\pi ni}{m}; \quad n = 0, \ldots, \frac{m}{2} - 1$$  \hspace{1cm} (19)

The recovered time function $f'(t)$ is represented as

$$f'(t) = \frac{a_n'}{2} + \sum_{n=1}^{m-1} \left( a_n' \cos \frac{2\pi ni}{m} + b_n' \sin \frac{2\pi ni}{m} \right)$$  \hspace{1cm} (20)

and the power spectrum is

$$p_n' = \sqrt{a_n'^2 + b_n'^2}$$  \hspace{1cm} (21)

Power spectrum truncation error

As discussed in “Truncation Errors in Sampling,” the spectrum error is found by taking the root mean square of the difference of the power spectra of the original and the recovered signals. The error of the cosine coefficients $E_{n\alpha}$ is

$$E_{n\alpha} = \sqrt{(a_n - a_n')^2} = \sqrt{\left( \frac{2}{m} \right)^2 \left[ \sum_{i=0}^{m-1} Y(iT) \cos \frac{2\pi ni}{m} \right]^2 - \left[ \sum_{i=0}^{m-1} X(iT) \cos \frac{2\pi ni}{m} \right]^2}$$

$$= \frac{2}{m} \sqrt{\sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2 \cos \frac{2\pi ni}{m}}$$  \hspace{1cm} (22)

using the Schwarz inequality. The upper bound of $E_{n\alpha}$ is

$$E_{n\alpha,\max} = \frac{2}{m} \sqrt{\sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2}$$  \hspace{1cm} (23)

since $\max (\cos \frac{2\pi ni}{m}) = 1$. Similarly, the error in the sine coefficients $E_{n\beta}$ is

$$E_{n\beta} = \frac{2}{m} \sqrt{\sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2 \sin \frac{2\pi ni}{m}}$$

and the upper bound of $E_{n\beta}$ is

$$E_{n\beta,\max} = \frac{2}{m} \sqrt{\sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2}$$  \hspace{1cm} (24)

The upper bound of the $jth$ error term in the power spectrum $E_{\rho n}$ is

$$E_{\rho j} = \sqrt{E_{n\alpha,\max}^2 + E_{n\beta,\max}^2} = \frac{2}{m} \sqrt{2 \sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2}$$  \hspace{1cm} (25)

The maximum total error of the power spectrum $E_{\rho}$ is

$$E_{\rho} = \sum_{j=0}^{N} \Delta f \rho_j = \frac{N \Delta f}{\rho} \sqrt{\sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2}$$  \hspace{1cm} (26)

where $N = m - 1$ and $\Delta f$ is the incremental frequency $\left( \frac{N}{m} \right)$. The total area under the power spectrum curve $a$ is

$$a = \sum_{j=0}^{N} \rho_j \Delta f$$  \hspace{1cm} (27)

The normalized power spectrum error $\overline{E}_{\rho}$ is obtained by taking the ratio of the total error in the power spectrum $E_{\rho}$ to the total area under the power spectrum curve $a$

$$\overline{E}_{\rho} = \frac{E_{\rho}}{a} = \frac{2 \sum_{j=0}^{N} \rho_j \sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2}{a}$$  \hspace{1cm} (28)

The normalized power spectrum error $\overline{E}_{\rho}$ is obtained by taking the ratio of the total error in the power spectrum $E_{\rho}$ to the total area under the power spectrum curve $a$

$$\overline{E}_{\rho} = \frac{\sum_{j=0}^{N} \rho_j \sum_{i=0}^{m-1} \left[ Y(iT) - X(iT) \right]^2}{a}$$  \hspace{1cm} (29)
The power spectrum error due to truncation $E_{\text{trunc}}$ is obtained by using equations (7) and (27)

$$E_{\text{trunc}} = \frac{2^{3/2} \Delta f}{m} \sum_{j=0}^{N} \sqrt{\sum_{i=0}^{m-1} \Delta_{\text{max}}^2}$$

$$= \frac{2^{3/2} \Delta f (N + 1)}{\sqrt{m}} \Delta_{\text{max}}^2$$

and the normalized power spectrum error due to truncation $E_{\text{trunc}}$ is obtained by using equation (29)

$$E_{\text{trunc}} = \frac{2^{3/2} \Delta f (N + 1)}{\sqrt{m}} \sum_{j=0}^{N} \frac{\rho_i}{2^{3/2} \Delta f (N + 1) 2^{a_i}}$$

and the normalized power spectrum error due to truncation $E_{\text{trunc}}$ is obtained by using equation (29)

$$E_{\text{trunc}} = \frac{2^{3/2} \Delta f (N + 1) 2^{a_i}}{\sqrt{m}} \sum_{j=0}^{N} \frac{\rho_i}{2^{3/2} \Delta f (N + 1) 2^{a_i}}$$

Power spectrum noise error

The error due to transmission noise is found in the same way that error due to noise and error due to truncation were found. The power spectrum noise error is obtained by using equation (27)

$$E_{\text{noise}} = \frac{2^{3/2} \Delta f}{m} \sum_{j=0}^{N} \sqrt{\sum_{i=0}^{m-1} (\Delta_{\text{max}} + \delta_{\text{max}})^2}$$

$$= \frac{2^{3/2} \Delta f (N + 1)}{\sqrt{m}} (\Delta_{\text{max}} + \delta_{\text{max}})$$

$$= \frac{2^{3/2} \Delta f (N + 1) [2^{a_i} + 2^{(k+1)q}]}{\sqrt{m}}$$

and the normalized power spectrum error $E_{\text{noise}}$ is obtained by equation (29)

$$E_{\text{noise}} = \frac{2^{3/2} (N + 1) [2^{a_i} + 2^{(k+1)q}]}{\sqrt{m}} \sum_{j=0}^{N} \frac{\rho_i}{2^{3/2} (N + 1) [2^{a_i} + 2^{(k+1)q}]}$$

COMPUTER SIMULATION OF DEGRADATION ANALYSIS

We now describe the results of a computer simulation which considers analog-to-digital conversion of the sampled signal, the corruption of the binary data by a random binary sequence, the digital-to-analog conversion (recovery) of the received binary signal, the computation of the sine and cosine coefficients of the recovered sampled signal, the computation of the power spectrum of the original and the recovered signals, and the computation of error. Included are the results of computer evaluations of the theoretical error bounds.

Figure 5 is a simplified flow graph of the simulation. Part 1 is concerned with storing on tape the transmitted and received sample values of the analog signal, while Part 2 is concerned with using the taped data and the determination of error.

Figure 6 contains a graph of the actual sampled analog signal. The analog signal was sampled at the rate of 20 samples/second, yielding 100 samples in all.

Figure 7 contains graph displays of $X$ and $Y$ for various values of $q$. For $q=0$, $10^{-4}$, $10^{-3}$, and $5 \times 10^{-3}$, there is little visible difference between the transmitted and the recovered sample values, $X$ and $Y$. From Figure 9 (corresponding to $q=10^{-3}$), the rms error between $X$ and $Y$ is about $10^{-2}$. For $q=10^{-3}$, the difference between $X$ and $Y$ becomes noticeable, as seen in Figure 7, sheets 4 and 5. From Figure 9 ($q=10^{-4}$), the rms error is greater than $10^{-2}$.

As shown in Figure 8 ($q=10^{-4}$), the difference between the original and the reconstructed power spectra becomes appreciably noticeable; and, as shown in Figure 10, the error between the spectra is greater than $7 \times 10^{-3}$.

Note that in both Figures 9 and 10, the error in the simulated case at times exceeds the theoretical limit by a magnitude of order of about 1. This is most likely due to the possibility that the random binary noise generator is generating slightly more noise within the relatively short sample period. On the average, if more runs were made, it is expected that the error would approach the theoretical limit. In any case, conclusions based upon the simulation would be conservative, since the error in the simulated case exceeds the theoretical error.

DISCUSSION AND CONCLUSIONS

We now present some results of degradation analysis of analog signals through a digital communication system.

Two measures of degradation were considered; i.e., (1) rms value of the difference between the transmitted sampled analog function and the received analog function, and (2) rms value of the difference between the spectra of the transmitted and received signals. Based upon the results just covered and assuming that an error of no more than 0.01 can be tolerated with either measure, an error rate of no more than 1 bit in 10,000 bits can be tolerated. Truncation error (the error due to limiting the digital equivalent of the sampled analog signal to 12 bits) is negligible.
In this analysis, the binary symmetric channel was used as the model of the digital communication system. The binary symmetric channel is a good model of a system which encounters independent uniformly distributed, random bit errors. For those systems whose bit errors are not independent, the burst noise or Gilbert channel should be considered. Through the use of proper encoding and error correction, it is possible to reduce effectively the burst noise channel to the binary symmetric channel. In the case of either channel, the above-stated error rate should be the goal of any error correction system.

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Figure 4—Transition diagram of the binary symmetrical channel

Figure 6—Analog time function of the Rayleigh fundamental mode

Figure 7—Original and noise-corrupted analog time functions of the Rayleigh fundamental mode

Figure 5—Simplified flow graph of computer simulation of a digital communication system

START

ENTER SAMPLED ANALOG SIGNAL AND STORE ON TAPE

DIGITIZED DATA (12 BITs/SAMPLE)

ADD INDEPENDENT, UNIFORMLY DISTRIBUTED RANDOM NOISE

CONVERT RECEIVED DIGITAL SIGNAL TO ANALOG AND STORE ON TAPE

READ FROM TAPE SAMPLED ANALOG SIGNAL AND RECONSTRUCTED ANALOG SIGNAL AT RECEIVER

COMPUTE AMPLITUDE, POWER SPECTRUM AND ERROR

COMPUTE ON AN RMS BASIS THE ERROR BETWEEN THE TWO SIGNALS

STOP

PART 1

PART 2
Figure 7 (Cont'd)

Figure 8—Original and noise-corrupted power spectra of an analog time function and the Rayleigh fundamental mode
Figure 8 (Cont'd)
Figure 9—Sampling error versus channel error probability

Figure 10—Power spectrum error versus channel error probability