A computer analysis of pressure and flow in a stenotic artery*

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INTRODUCTION
There is a widespread assumption in the medical and engineering communities that when a small decrease in cross-sectional area is imposed upon an artery there is a correspondingly small decrease in arterial flow and pressure, and that when progressively larger increases in arterial stenosis are made there is a correspondingly greater decrease in arterial pressure and flow. That this is not so has been shown in earlier studies$^1,2$ in a qualitative way. When a small stenosis is imposed upon an artery there is no change in arterial pressure and flow until a rather large and significant stenosis is produced. Recent studies$^3,4$ have attempted to develop this concept even further. It has been the object of this study to quantitate the effect on the pressure and flow in an artery that is subjected to an increasing stenosis. This has been done by drawing from a background of numerous arterial flow and pressure determinations in both humans and dogs, and then carrying this further to the development of a mathematical model and subjecting the mathematical model to computer analysis.

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Materials and methods
The basis for the construction of the mathematical model was determined in the following manner: Numerous blood flow measurements were made with a square-wave electromagnetic blood flowmeter.* Pressure measurements were made through No. 20-gage needles inserted into the artery and connected to short lengths of rigid wall nylon tubing which, in turn, were connected to strain gages.** These were connected to strain gage pre-amplifiers† and displayed on a suitable oscilloscope.‡ The data were recorded on a 7-channel instrumentation tape recorder§ arranged in the FM mode of recording with a frequency response flat from 0 to 2500 Hz. Photographs were made of the analog oscilloscope display by using a Tektronix Model C-12 Polaroid Camera that can take photographs directly from the face of the oscilloscope tube. These data were then subjected to qualitative analysis, reduced manually to digital form, and the mathematical model which is described here was developed. The requisite mathematical

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*Avionics Research Products, Model MS-6000A.
**Statham Model P-23Db.
†Tektronix Model Q Oscilloscope Plug-in.
‡Tektronix Model RM-335A Oscilloscope.
§Ampex Model SP-300 Instrumentation Tape Recorder.
equations were solved and the results plotted out on a Digital Computer.§§ The Computer Block Diagram is presented in Figure 1.

![Computer block diagram for computation of distal flow and pressure in a stenotic artery](image)

**Figure 1**—Computer block diagram for computation of distal flow and pressure in a stenotic artery

**Mathematical model**

In developing a mathematical model of the portion of an artery in stenosis, we will idealize the artery to the extent of considering only rigid wall tubing of constant cross-section, except in the immediate area of the stenosis. This idealized system is shown in Figure 2.

![Schematic representation of an idealized stenosis in an artery](image)

**Figure 2**—Schematic representation of an idealized stenosis in an artery

A momentum balance between points 1 and 2 and points 4 and 5 yields:

\[
\frac{dQ_{in}}{dt} = \frac{k_1A_0}{pL} (P_1 - P_2) - \frac{RQ_{in}}{pA_0} \tag{1}
\]

\[
\frac{dQ_{out}}{dt} = \frac{k_1A_0}{pL} (P_4 - P_5) - \frac{RQ_{out}}{pA_0} \tag{2}
\]

where \(Q\) is the volumetric flow rate, \(R\) is the resistance to flow, and \(p\) is the density of blood. Under these conditions, the continuity equation takes the form:

\[Q_{out} = Q_{in} (p = \text{constant}) \tag{3}\]

therefore, by virtue of equation (3) we make equate equations (1) and (2):

\[
\frac{dQ_{in}}{dt} = \frac{dQ_{out}}{dt}; \text{or} \tag{4}
\]

\[P_1 - P_2 = P_4 - P_5 \]

Application of the Steady State Bernoulli equation (i.e., Conservation of Energy) to section 2, 3 in Figure 2 yields:

\[\Delta(1/2v^2/k_f) + \frac{\Delta P}{p} + 1/2 (v_2)^2 e_v = 0 \tag{5}\]

where \(v\) is the linear velocity, \(e_v\) is the total energy loss in the section under consideration, and \(k_f\) is a conversion factor. Then:

\[\frac{v_2^2}{2k_f} - \frac{E_v^2}{2k_f} + \frac{P_3 - P_2}{p} + 1/2 \frac{v_2^2}{k_f} e_v = 0 \tag{6}\]

We may evaluate \(e_v\) for a sudden contraction\(^2\) as:

\[e_v = .45 (1 - B) \tag{7}\]

where

\[B = A/A_0; \text{ thus,} \]

\[v_3A = vA_0\]

or

\[v_3 = v \frac{A_0}{A} = v/B\]

Therefore, (6) becomes:

\[\frac{v^2}{2B^2k_f} - \frac{v^2}{2k_f} + \frac{P_3 - P_2}{p} + 1/2 \frac{v^2}{B^2k_f} (.45) (1 - B) = 0 \tag{8}\]

or

\[P_3 = -\frac{pv^2}{2k_f} \left[1/B^2 - 1 + \frac{.45}{B^2} (1 - B)\right] + P_2 \tag{9}\]
 Similarly, applying Bernoulli’s Equation to sections 3, 4 in Figure 2 yields:

\[ P_3 = \frac{F}{2k_1} v^2 [1 - 1/B^2 + (1/B - 1)^2] + P_4 \]

(10)

since \( e_x \) for a sudden expansion is \( e_x = (1/B - 1)^2 \)

Equating equations (9) and (10) gives:

\[ P_2 = P_4 + \frac{Fv^2}{2k_1} [1 + (1/B - 1)^2 + .45 (1 - B)] \]

(12)

We may substitute the continuity relationship:

\[ P_1 - P_2 = P_4 - P_5 \]

into equation (12) to give:

\[ P_5 = P_1 + \frac{P_5}{2} + \frac{Fv^2}{2k_1} [(1 - B)^2 + .45 (1 - B)] \]

(13)

Substituting (13) into equation (1) yields:

\[
\frac{dQ_{out}}{dt} = \frac{k_1A_0}{pL} \left[ \frac{P_1 - P_5}{2} - \frac{Fv^2}{4k_1B^2} \right]
\]

\[ \left[ (1 - B)^2 + .45 (1 - B) \right] - \frac{RQ_{out}}{pA} \]

(14)

Now, analysis of the actual experimental data and the numerical values of the parameters in equation (14) implies that there is no phase-lag in the pressure-flow relationship, or:

\[ \frac{dQ_{out}}{dt} = 0 \]

Thus, we finally obtain from equation (14) the desired relationship:

\[ Q^2 \left[ \frac{(1 - B)^2 + .45 (1 - B)}{2B^2 + .45 (1 - B)} \right] 
+ \frac{2RLQ}{p} - \frac{k_1A_0^2}{p} (P_1 - P_3) = 0 \]

(15)

from which the flow rate, Q, may be determined.

We may also determine the distal pressure, \( P_4 \), by considering equation (2) and noting that

\[ \frac{dQ_{out}}{dt} = 0 \]

or

\[ \frac{dQ_{out}}{dt} = 0 = \frac{k_1A_0}{pL} (P_4 - P_3) - \frac{RQ_{out}}{pA} \]

(16)

Therefore

\[ P_4 = P_5 + \frac{RL}{k_1A_0^2} Q_{out} \]

(17)

![Graph](from the collection of the Computer History Museum (www.computerhistory.org))

**Results and discussion**

Figure 3 presents the solutions to equations (15) and (17) as plots of the distal flow, as a percentage of the normal-unobstructed-flow, and the distal pressure difference, \( P_4 - P_5 \), as a percentage of the normal pressure difference, against increasing stenosis in the artery.

Figure 4 is a cross-plot of the percentage of unobstructed distal flow against the percentage of unobstructed distal pressure differential, with stenosis as an implicit parameter. Note from Figure 3 the point of critical stenosis at 75 per cent (i.e., where flow and pressure drop precipitously), and from Figure 4 the expected linearity of the pressure-flow relationship, independent of stenosis, as predicted by equation (15).

Figure 5, a plot of flow and pressure against steno-
The very reason for this error is that in such situations the elasticity of the arterial walls must be considered. The effect of elasticity will cause the continuity relation:

\[ Q_{\text{out}} = Q_{\text{in}} \]

to break down. Subsequent work will show how the effect of elasticity at large stenosis may be determined. This will lead to a method for determining arterial elastance under actual flow conditions by simple measurements of flow and pressure.

**SUMMARY**

It had been a previous, widespread assumption that when an artery is subjected to a gradual, progressive, stenosis there was an immediate, gradual, progressive, fall in arterial blood pressure and flow, i.e., the relationship was linear. Experimental work over the past seven years has shown that this is not so; the relationship is not linear. Experimental observations in the human and on the dog have shown that when an artery is subjected to a decreasing cross-sectional area, as produced by a stenosis, the flow through that artery remains unchanged until a severe decrease in cross-sectional area is attained. Prior to this point there is no change in blood flow, i.e., it remains normal. When this point of critical stenosis is reached, there is a sudden, marked, and precipitous drop in blood flow through the artery.

Application of the principles of conservation of energy and momentum to the system under consideration have led to a mathematical system which accurately describes the phenomenon. A computer program has been written which describes this system. The print-out of the computer results is presented in Table 1.

It is further expected that this mathematical system will enable one to determine experimentally the
From the collection of the Computer History Museum (www.computerhistory.org)

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CONCLUSIONS
Numerous determinations of arterial flow and pressure have been made in the human and in the dog in normal arteries and in arteries that have been subjected to stenosis, either by arteriosclerosis or by physical stenosis produced by a circumferential ligature about an artery. Data collected from this source have shown that when a stenosis is imposed upon an artery, there is initially no change in pressure or flow within that artery, proximal to the stenosis, until a certain critical point is reached where arterial flow and pressure simultaneously fall precipitously. This point of critical arterial stenosis is different for different arteries.

Drawing upon a large experimental background, a mathematical model has been developed to define the effect of a gradual and progressive arterial stenosis upon arterial flow and pressure.

Having once defined the mathematical model, the computer model was developed. The computer results were shown to be identical with the actual measurements made on the human and on the dog.

A recheck of the computer model results was then done on the dog in several arteries, and the results in the in vivo preparation corroborated the computer results.

A mathematical and computer model have been developed as an approach to the study of the effect upon arterial pressure and flow when that artery is subjected to a progressive stenosis.

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**REFERENCES**

1. **F. C. MANN, J. F. HERRICK, H. ESSEX, E. J. BALDES**
   - The effect on the blood flow of decreasing the lumen of a blood vessel
   - Surgery 4:249-252 1938

2. **T. C. GUPTA, C. J. WIGGERS**
   - Basic hemodynamic changes produced by aortic coarctation of different degrees
   - Circulation 3:17-31 1951

3. **A. G. MAY, J. A. DE WEESE, C. G. ROB**
   - Hemodynamic effects of arterial stenosis
   - Surgery 53:513-524 1963

   - Critical arterial stenosis
   - Surgery 54:250-259 1963

5. **R. B. BIRD, W. STEWART, E. N. LIGHTFOOT**
   - Transport Phenomena
   - John Wiley & Sons Chapter 8 New York 1960

6. **R. B. BIRD, W. STEWART, E. N. LIGHTFOOT**
   - Transport Phenomena
   - John Wiley & Sons Chapter 8 New York 1960

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