INTRODUCTION

Multi-variable problems are among the most difficult types of problems for an electronic analog computer to solve, for such a computer is a single-independent-variable machine. Approximations are invariably made and the mathematics used is frequently borrowed from the domain of numerical analysis. The area of multiple integrals is no exception. A typical numerical technique for evaluating an integral of the type

\[ I = \int_a^b \int_{x_i}^{x_{i+1}} f(x, y) \, dy \, dx \]  

is to divide the interval from \(a\) to \(b\) into \(n\) intervals \(\Delta x_i\); as in Figure 1. Then

\[ I \approx I_A = \sum_{i=1}^{n} \Delta x_i \int_{x_i}^{x_{i+1}} f(x, y) \, dy \]  

where \(x_i\) is the mean value of \(x\) in the interval \(\Delta x_i\). It is well known that \(I_A \to I\) when \(n \to \infty\) and \(\max (\Delta x_i) \to 0\). Generally, it is easiest to compute the approximation of \(I\) by taking \(\Delta x_i\) constant. Then, \(\Delta x_i = \Delta x\), and

\[ n \Delta x = |a - b|. \]

For a sufficiently large \(n\), a reasonable approximation is obtained. To minimize \(n\), various quadrature formulas\(^1\) are available. Rubin, Laudauer, and Totten\(^2\) used this technique with a 16-point Gaussian quadrature formula in the evaluation of a set of six double integrals involved in antenna pattern calculations on an analog computer. This method affords a reasonable compromise between accuracy and time of computation, as well as reducing the amount of equipment required. (All functions of \(x\) are constants during the calculation in each interval.)

The above method is essentially a digital one, and would become tedious on an analog com-
computer for integrals of order higher than two. Rogers and Connolly suggest the use of a scanning technique for higher-order integrals. Each variable is forced to change continually in a sawtooth fashion. No details are given in their book.

This paper develops techniques for solving (1) that allow \( x \) to vary continually from \( a \) to \( b \) as \( y \) oscillates quickly between \( y_2 \) and \( y_1 \). The scan is a continuous path and affords an automatic and convenient method of analog computation. The formulas derived do not require the path to be linear, although such a path is by far the most convenient. Both diode and relay circuits are given, as well as two illustrative examples of how the circuits are used. Extensions to higher-order integrals are also made, the accuracy of computation diminishing as the order of the integral increases.

When a repetitive computer with memory is used, a suggested method for calculating multiple integrals introduces an error due to mathematical approximation, even in limiting cases such as \( \frac{df}{dy} = 0 \). The techniques used here introduce no approximation error for such a situation. One easily concludes that the methods introduced here are preferable even when a repetitive computer with memory is available.

MATHEMATICAL APPROXIMATIONS

By distorting the path of integration to allow for a continually increasing \( x \), as in Figure 2, the following approximation to \( I \) holds:

\[
I \approx I_A = \sum_{i=1}^{n} \Delta x_i \int_{y_i(x)}^{y_{i+1}(x)} f(x, y) \, dy
\]

where \( \Delta x_i = x_{i+1} - x_i \), and \( x \) in this interval is a single-valued function of \( y \) but not necessarily linear. The integration is performed with respect to \( |dy| \) to allow for a descending path from \( y_i \) to \( y_1 \), without changing the sign. (We assume \( y_2(x) \geq y_1(x) \).) It follows that as \( \max(\Delta x_i) \to 0 \) and \( n \to \infty \), then \( I_A \to I \).

For the purposes of analog computation, \( y \) is made a single-valued function of \( x \) and is forced to oscillate between \( y_2(x) \) and \( y_1(x) \). Mathematically, we seek a \( y \) such that

\[
y = (y_2 - y_1) \varphi(x) + y_1
\]

where \( \varphi(x) \) is a continuous function of \( x \) that oscillates between 0 and 1, with bounded derivatives. \( \varphi(a) \) can be either 0 or 1, and it is best that \( \varphi(b) \) be either 0 or 1, to eliminate any partial intervals. Then

\[
dy dx = \frac{dy_1}{dx} + \left[ \frac{dy_2}{dx} - \frac{dy_1}{dx} \right] \varphi(x) + (y_2 - y_1) \frac{d\Omega(x)}{dx}.
\]

The frequency of oscillation is required to be high so that the dominant term of (6) will be

\[
(y_2 - y_1) \frac{d\Omega(x)}{dx}.
\]

Equations (6) reduces to

\[
dy dx = (y_2 - y_1) \frac{d\Omega(x)}{dx} = (y_2 - y_1) \varphi(x)
\]

\[
\text{and}
\]

\[
|dy| = (y_2 - y_1) |\varphi(x)| dx.
\]

Equation (4) reduces to

\[
I \approx \sum_{i=1}^{n} \Delta x_i \int_{x_i}^{x_{i+1}} f(x, y) (y_2 - y_1) \varphi(x) \, dx.
\]

Equation (9) is the basis for making \( I \) the result of a continuous integration of a single variable.

THE CHOICE OF \( \varphi(x) \)

Of the more common oscillatory patterns, the triangular wave offers the advantage of simplicity. Sinusoidal waves have been considered, but introduce additional multiplications and...
additional equipment. Their only advantage is to eliminate switching problems which occur when triangular waves are considered. Hence, we let 

\[ \Omega(x) = k(x - x_i) \text{ for } x_i \leq x \leq x_{i+1} \]  
and 
\[ = 1 - k(x - x_{i+1}) \text{ for } x_{i+1} \leq x \leq x_{i+2} \]  
for \( k > 0 \) and odd \( i \). Since \( x_1 = a \), we have chosen \( \Omega(a) = 0 \) without loss of generality. The function \( y \) for two successive intervals becomes 

\[ y = y_1 + k(y_2 - y_1)(x - x_i) \]  
and 
\[ y = y_2 - k(y_2 - y_1)(x - x_{i+1}) \]  
for odd \( i \).

In each interval, \( \Omega(x) \) changes by one unit (from 0 to 1 or 1 to 0), so that from (10) 

\[ 1 = k \Delta x_i. \]  
Since 
\[ |\Omega'(x)| = k, \]  
the integral becomes 

\[ I = \sum_{i=1}^{n} \frac{1}{k} \int_{x_i}^{x_{i+1}} k f(x, y) (y_2 - y_1) \, dx \]
\[ = \int_{a}^{b} f(x, y) (y_2 - y_1) \, dx. \]  

Note that the slope \( k \) has dropped out of (14) before summation so that the approximation holds if the area bounded by \( a \leq x \leq b \) and \( y_1 \leq y \leq y_2 \) is traversed by straight lines of any slope. The slope \( k \) may be permitted to change slowly and vary from interval to interval, an important concept when consideration of generating (11) on the computer is made. Since \( y_2 - y_1 \) will be, after proper scaling, a slowly changing function of \( x \), for two successive intervals (11) is amended to be 

\[ y = y_1 + c(x)(x - x_i) \]  
and 
\[ y = y_2 - c(x)(x - x_{i+1}) \]  
where \( c(x) \) any positive function of \( x \) which is roughly constant in any interval*.

The equations to be solved, from (14) and (15) are 

\[ \frac{dI}{dx} = [y_2(x) - y_1(x)] f(x, y) \]  
and 
\[ \frac{dy}{dx} = \pm c(x). \]  

* This formulation accommodates the parametric technique for avoiding division whenever denominators contain function of \( x \).

In differentiating (14), \( b \) is replaced by \( x \). It is understood that the computation ends when \( x = b \). In differentiating (15) to obtain (17), it is assumed that \( y_1, y_2, \) and \( c(x) \) are constants, or that 

\[ \frac{dy_1(x)}{dx}, \frac{dy_2(x)}{dx} \text{, and } \frac{dc(x)}{dx} \]

are small compared to \( c(x) \).

MECHANIZATION ON THE COMPUTER

Three distinct problems are present in the solution of (16) and (17) by an analog computer:

1. Generation of \( y_2(x) \) and \( y_1(x) \)
2. Generation of \( y \)
3. Generation of \( f(x, y) \)

We shall be concerned only with the generation of \( y \). The techniques and difficulties of generating functions of one or more variables are discussed in virtually every text on analog computation. This paper can only be applied if all functions required by (16) are generable.

The function \( c(x) \) is virtually arbitrary and is selected for convenience. The problem involved in mechanizing (15) is to switch the sign of the input to the integrator producing \( y \) when \( y \) reaches either \( y_2 \) or \( y_1 \). One method of doing this is illustrated in Figure 3 with \( c(x) \) proportional to \( y_2(x) - y_1(x) \). The bang-bang circuit is strongly stable and diode-limited. That \( y \) remains between \( y_2 \) and \( y_1 \) can be observed by tracing the output of the high-gain amplifier when the summer output is both positive and negative. Figure 4 gives an equivalent circuit using a high-speed relay for switching. Note

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that the frequency potentiometer in both cases serves only to change the number of intervals and has no bearing in the solution of (16) and (17). However, the approximation (4) is improved when \( n \) is large. It is desirable, therefore, to obtain as high an input rate as possible.

One obvious method of getting the largest possible number of intervals is to let \( c(x) \) be constant, for it may then be scaled to 100 \( v \). When diodes are used for switching in this case (fig. 5) more equipment is required than when relays are used (fig. 6). Which circuit to use is dependent upon the type of equipment available. If high speed double-pole double-throw switches are available, the circuit shown in figure 6 uses a minimum of amplifiers and undoubtedly provides the greatest accuracy.

The rest of the solution of (16) and (17) is straightforward. Since \( y_1(x) - y_1(x) \) is a relatively slowly changing variable, it may be used to drive the shaft of a servo multiplier. The function \( f(x, y) \) must be generated with high-speed equipment, as \( y \) will be a rapidly changing function of time.

**ERROR AND ACCURACY**

It is necessary to distinguish between approximation error and computation error, even though they are not mutually exclusive. The limitation of the amplifiers in reproducing high frequencies limits the number of intervals into which the interval from \( a \) to \( b \) may be decomposed. An effort to decrease the approximation error by requiring a higher frequency of oscillation may very well decrease the net accuracy due to a substantial increase in the computation error. A case in point is the example in the next section where a change in the number of intervals from 30 to about 800 does not materially improve the net accuracy. One may generally assume that the approximation error is negligible (by analog computer standards) unless there are relatively high-frequency components in \( f(x, y) \), \( y_2 \), or \( y_1 \).

No approximation error exists in this technique when the integrand of the double integral is a function of \( x \) only:

\[
\int_{a}^{b} \int_{y_1(x)}^{y_2(x)} f(x, y) \, dy \, dx = \int_{x_1}^{x_2} f(x) \, g(x) \, dy \, dx \quad (18)
\]
The approximation expressed by (14) is then exact. One would not use multiple integral techniques for such a situation, but this example serves well to show qualitatively why the techniques used here are preferable to those suggested when a repetitive computer with memory is available. The basic scheme suggested is to compute, store, and update the inner integral
\[ \int_{x_1}^{x_2} f(x, y) \, dy \]
at a high repetitive rate as \( x \) is continually increasing. The output of the memory is a staircase function of \( x \) and is integrated to obtain the required double integral. Even in the trivial case given in (18), one is required to use a high repetitive rate in order to minimize the approximation error. This is due only to the staircase function which appears at the output of the memory. One may conclude generally that techniques involving memory introduce more approximation error for an evaluation of a multiple integral than the techniques introduced in this paper.

It was determined experimentally that computation error increased greatly when changes in \( y \) exceeded 5000 \text{v/sec} for all examples tried*. This upper limit would be greater if the frequency responses of the equipment were improved. Besides normal computing errors, non-ideal diodes and relays contribute to the computation error. For a 5000 \text{v/sec} excursion for \( y \), 100 \text{v} is attained in 20ms. Relays are inapplicable unless switching times are in the microsecond ranges. Simple diode circuits, on the other hand, will not give a constant output. Not only do \( y \) deviate from its linear relationship with \( x \), but also the switching occurs at values other than \( y_2 \) and \( y_1 \). The use of idealized diode circuits, however, completely eliminates these errors although they require more equipment. Langill's multivibrator circuit may be modified to satisfy the requirements on \( y \) and produce accurate diode switching.

The accuracy with which all the test double integrals were evaluated was better than one percent, with many solutions giving better than 0.1 percent accuracy, when using the simple diode circuits shown in this paper.

EXAMPLE

For demonstration, we take
\[ I = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} 2\pi y \, dy \, dx = 4\pi \int_{0}^{\pi} y \sqrt{1-x^2} \, dx \]
which represents the volume of a sphere of radius 1. By (14) we use
\[ I \approx 2\pi \int_{-1}^{1} y \sqrt{1-x^2} \, dx = 4\pi \int_{0}^{\pi} y \sqrt{1-x^2} \, dx \]
with the understanding that \( y \) is generated by any one of the circuits shown in figs. 3-6. A computer diagram, shown in Figure 7, utilizes a servo multiplier for finding the square root and for multiplication. Note that the output of the integrator representing \( I \) is in reality
\[ I(x) \approx 4\pi \int_{0}^{\pi} y \sqrt{1-x^2} \, dx . \]

* An Electronics Associates, Inc., PACE 131-R Analog Computer was used. Tests were made with simple diode limiting circuits because no fast-acting relays were available.

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guaranteeing a minimum of 20 intervals*. Note that the size of the intervals decreases as \( y_2 \) approaches \( y_1 \) whenever the slopes are constant, as in this case.

Figure 9 shows the graph of \( I(x) \) obtained. Note that the curve oscillates about the true answer which is superimposed on the graph:

\[
I(x) = 4\pi \int_0^x \sqrt{1-x^2} \, y \, dy \, dx = 2\pi \left[ x - \frac{x^3}{3} \right].
\]  

(22)

The results are most accurate at the end of each interval.

In order to increase the slopes and shorten computation time, the diode circuit for producing \( y \) as shown in figure 5 was used next. The computation time was shortened to 10 seconds, and 5000 v/sec was obtained for \( y \) by using a 0.1\( \mu \)F feedback capacitor with a 0.2M\( \Omega \) input resistor. This produced slopes of \( \pm 500 \) and about 800 intervals. Because the oscillations are small, \( I(x) \) as shown in Figure 10, appears as a relatively smooth curve.

\[
I(x) = 4\pi \int_0^x \sqrt{1-x^2} \, y \, dy \, dx
\]  

Using Slopes of \( \pm 500 \).

**EXTENSIONS TO HIGHER-ORDER INTEGRALS**

Integrals of the form

\[
I = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x, y)}^{z_2(x, y)} dx \, dy \, dz
\]

are amenable to analysis similar to that which has been applied to integrals of type (1). Formulas similar to those of (14) are obtained if straight line paths are used:

\[
I = \int_a^b (y_2 - y_1)(z_2 - z_1)f(x, y, z) \, dx
\]

(23)

where

- \( z \) oscillates between \( z_2 \) and \( z_1 \), at the highest frequency;
- \( y \) oscillates between \( y_2 \) and \( y_1 \), at a medium frequency; and
- \( x \) traverses from \( a \) to \( b \).

In general, one requires \( n \)-1 different frequencies of oscillation for an \( n \)-th-order integral. The slowest variable does not oscillate but traverses from one end of its limit to the other. The accuracy of the computation is probably great-
est if the ratio of successive frequencies of oscillation are equal. Thus the rate at which the variables change should satisfy a geometric progression. If the slowest variable is moved at \( p \) v/sec and the fastest variable at \( q \) v/sec, the ratio \( r \) of successive voltage rates satisfies

\[ q = pr^{n-1}. \tag{25} \]

The \( i^{th} \) intermediate voltage rate \( R_i \) is then given by

\[ n_i = pr^i = p \left( \frac{q}{p} \right) \frac{i}{n-1}. \tag{26} \]

For the PACE 131-R Analog Computer used in this study, a third-order integral would have voltage rates of 1, 70, and 5000 v/sec. A typical flow diagram for the solution of (23) is shown in Figure 11. One may decrease \( p \) to increase the ratio of voltage rates, but integrator drift rates prevent too low a limit. If the independent variable traverses 100 v during the computation, then the computing time is 100 sec. Decreasing the computing time is accomplished with a sacrifice of accuracy. A change of voltage rates to 5, 160, and 5000 v/sec decreases the computing time to 20 sec for a third-order integral, but cuts the number of volume elements by the same factor of 5, and the approximation error increases. Fifth-order test integrals have been evaluated with reasonable success with voltage rates of 0.5, 5, 50, 500, and 5000 v/sec. In general, one can expect the accuracy to diminish quite severely for higher-order integrals unless sufficiently large voltage ratios are obtainable. With the same computer, approximate accuracy limits for a few sample integrals were 1, 2, 5, and 10 percent for second-, third-, fourth-, and fifth-order integrals. Usually, the results were better, but this was highly dependent on the problem.

**ANOTHER APPLICATION**

If a simple integral contains a parameter \( x \)

\[ I(x) = \int_{a(x)}^{b(x)} f(x, y) \, dy \tag{27} \]

then

\[ \frac{dI}{dx} = \int_{a(x)}^{b(x)} \frac{\partial f(x, y)}{\partial x} \, dy + f(x, b) \frac{db}{dx} - f(x, a) \frac{da}{dx} \tag{28} \]

from which

\[ I(x) = \int_a^b \int_{a(x)}^{b(x)} \frac{\partial f(x, y)}{\partial x} \, dy \, dx + \int_a^b \left[ f(x, b) \frac{db}{dx} - f(x, a) \frac{da}{dx} \right] \, dx + I(x_0). \tag{29} \]

To apply this technique \( I(x_0) \) must be known or calculated for \( x = x_0 \) for use as an initial condition. In addition, the derivatives as indicated in (29) must be generable. The procedure is not unlike the generalized integration method.

When the frequency of oscillation of \( y \) is large, the amplifier output which represents the double integral appears as a relatively smooth curve, so that (27) is reasonably represented by (29).

For example

\[ I(x) = \int_0^x e^{-xy} \, dy \tag{30} \]

yields

\[ I'(x) = \int_0^x y e^{-xy} \, dy + e^{-x} \tag{31} \]

and

\[ I(x) = \int_0^x \int_0^y e^{-xy} \, dy \, dx + \int_0^x e^{-x} \, dx \cdot \tag{32} \]
Figure 12 shows a graph of \(30\) as approximated by changing \(32\) to

\[
I(x) = \int_0^x \left[ e^{-x} - x y e^{-x} \right] dx \quad (33)
\]

and generating \(y\) with the diode circuit of figure 5 with slopes of \(\pm 500\). The term \(x y^2\) was formed by an electronic multiplier, \(e^{-x} y\) by a diode function generator and \(e^{-x^3}\) by a generalized integration process.

CONCLUSIONS

Multiple integrals are easily evaluated with non-repetitive analog computers and components generally available therein. The approximation formulas derived are inherently more accurate than those used with techniques involving memory. No approximation error is accumulated if \(f(x,y)\) in (1) is equal to 1, and the integral represents the area bounded by \(a, b, y_2(x)\) and \(y_1(x)\). In fact, (14) is exact if \(f(x,y)\) does not contain \(y\). Therefore these techniques are preferable even for repetitive computers with memory.

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