ANALOG SIMULATION OF THE RE-ENTRY OF A BALLISTIC MISSILE WARHEAD
AND MULTIPLE DECOYS

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Summary

The basic problem considered here is the computation of the reentry trajectory of a single ballistic missile warhead as well as the trajectories of a number of decoys which originate from the warhead trajectory. Suitable three-dimensional equations of motion are presented for a reentry vehicle with arbitrary drag coefficient, mass, and area, and the analog computer circuit for solving these equations in real time is given. Then a method of using several such circuits to compute simultaneously the trajectories of multiple targets with variations in all three initial velocity components as well as variations in ballistic coefficient is presented.

Introduction

One of the more interesting current problems in simulation involves the computation of the trajectories of a very large number of reentry vehicles, each starting with the same position coordinates and approximately the same velocity coordinates, but with the possibility of widely varying ballistic coefficients. For example, this is the problem presented in the simultaneous simulation of a reentering ballistic missile warhead and a large number of decoys. In this paper we will show first how each trajectory can be computed in real time using a modest amount of analog computing equipment. If we compute decoy trajectories for the extreme upper and lower limits of velocity perturbations and ballistic coefficient variations, then simple linear or second order interpolation can be used to obtain the trajectory for any decoy with intermediate velocity or ballistic coefficient variation. The required averaging and summing operations can be accomplished using simple passive resistors. The net result is that a reasonable amount of analog equipment, as needed to generate the limiting trajectories, can be used as a basis for simultaneous generation of hundreds of decoy trajectories.

First we will consider the appropriate equations of motion for a single reentering particle. The analog-computer solution of these equations will then be presented using a mechanization with solid-state nonlinear components. This adds the possibility of high speed repetitive operation, although for simulation purposes real time computation would be more appropriate. Finally, the technique of generating a very large number of trajectories from several limiting trajectories will be described.

Equations of Motion in the Trajectory Plane

It will be assumed that for purposes of this simulation the rotation of the earth can be neglected. Although this assumption is not in any way necessary to implement a practical analog solution, it simplifies the equations and hence the computer mechanization of the problem. The actual errors in trajectory as viewed from a position near the impact point as a result of this assumption will be quite small compared with the total trajectory distance traversed in the simulation.

With the above assumption the trajectory for a pure drag vehicle will lie in a vertical plane through the center of the earth. We can use polar coordinates $\rho, \theta$ in this plane to describe the position of the warhead, which we will assume is a point mass $m$ (see Figure 1). We denote the horizontal velocity component by $U_h$ and the vertical velocity component by $W_h$ (positive downward). Summing forces in the horizontal and vertical directions, we obtain

\begin{align}
    m(\ddot{\rho} + \dot{\theta}^2) &= X_h \\
    m(\ddot{\rho} - \dot{\theta}^2) &= -\frac{mg_0 r_0^2}{r^2} - Z_h
\end{align}

Here $X_h$ and $Z_h$ are horizontal and vertical forces due to aerodynamic drag, and $g_0$ is the acceleration due to gravity at a fixed distance, $r_0$, from the center of the spherical earth. The term $2\dot{\theta}$ in Eq. (1) is the Coriolis acceleration, whereas $\dot{\theta}^2$ in Eq. (2) is the centrifugal acceleration.

The terms $\ddot{\rho} + 2\dot{\theta}$ in Eq. (1) can be rewritten as $1/r \, d/dt(r^2 \dot{\theta})$. Thus the equation can be...
integrated to give

\[ \int r X_h \, d\tau + \frac{mr^2 \theta}{r} \left|_{r=0}^{t} \right. \]

This is the well known angular momentum integral. We note that

\[ U_h = r \dot{\theta}, \quad W_h = -\dot{r} \]

Thus Eqs. (2) and (3) can be rewritten as

\[ W_h = \frac{t}{r^2} \left( \frac{\theta}{r^2} - \frac{U_h^2}{r} + \frac{Z_h}{m} \right) \, d\tau + W_h \left|_{t=0} \right. \]

and

\[ U_h = \frac{1}{r} \int \frac{X_h}{m} \, d\tau + \frac{rU_h}{r} \left|_{t=0} \right. \]

The aerodynamic drag force \( D \) will be directed opposite to the warhead total velocity vector, as shown in Figure 2, and is given by

\[ D = \frac{1}{2} \rho_a V_a^2 C_D A \quad (7) \]

where \( \rho_a \) is the atmospheric density, \( V_a \) is the total velocity, \( C_D \) is the drag coefficient, and \( A \) is the characteristic area on which the drag coefficient is based. \( C_D \) will in general be a function of Mach number \( M \), although it may be sufficiently accurate for purposes of this simulation to assume that \( C_D \) is constant. The drag components \( X_h \) and \( Z_h \) along the horizontal and vertical (downward) directions are given, respectively, by

\[ X_h = -D \frac{U_h}{V_a} = -(\frac{1}{2} \rho_a C_D A) \frac{U_h}{V_a} \]

\[ Z_h = -D \frac{W_h}{V_a} = -(\frac{1}{2} \rho_a C_D A) \frac{W_h}{V_a} \]

where the total velocity \( V_a \) is given by

\[ V_a = (U_h^2 + W_h^2)^{1/2} \]

Finally, we will assume an exponential atmospheric model. Thus let

\[ \rho_a = \rho_{ao} e^{-h/h_o} \]

where the altitude \( h \) is given by

\[ h = r - R \]

\( R \) is the radius of the spherical earth. Eqs. (5), (6), and (8) through (12) are the basic equations for motion in the plane of the trajectory. Scaling of these equations for the computer is simplified if we introduce a dimensionless radial distance \( \rho \) and a perturbation \( \delta \rho \) given by the formulas

\[ \rho = \frac{r}{r_o}, \quad \delta \rho = \rho - 1 \]

From equation (5) with \( W_o = W_h \left|_{t=0} \right. = Z_h = X_h = 0 \) we note that a satellite in a zero-drag circular orbit at radial distance \( r_o \) will have a velocity \( U_{ho} \) given by

\[ U_{ho} = \sqrt{\frac{r_o \Theta_o}{\rho}} \]

It is convenient to define dimensionless velocities in terms of \( U_{ho} \). Thus let

\[ u_h = \frac{U_h}{U_{ho}}, \quad w_h = \frac{W_h}{U_{ho}}, \quad v_h = \frac{V_h}{U_{ho}} \]

In terms of the dimensionless variables of Eqs. (13) and (15), Eqs. (5) and (6) become

\[ w_h = \frac{t}{r_o} \left( \frac{1}{1 + \delta \rho} - u_h^2 + \frac{Z_h}{mg_o} \right) \, d\tau + \frac{w_h}{t=0} \]

and

\[ t \frac{\rho}{U_h} \left( 1 + \delta \rho \right) \left[ \frac{X_h}{mg_o} \right] \, d\tau + \frac{v_h}{t=0} \]

where \( \rho_o \) and \( u_{ho} \) are the initial values of \( \rho \) and \( u_h \) and where \( T \) is a time constant equal to the reciprocal of the circular-orbit frequency and is given by

\[ T = \frac{1}{\sqrt{r_o \Theta_o}} \]

For typical ballistic reentry trajectories the altitude at initiation of trajectory simulation may range as high as 200 statute miles. If we choose 100 statute miles as the altitude of the circular reference orbit, then \( r_o = 3950 + 100 = 4050 \) statute miles and \( r \) ranges between 4500 miles initially and 3950 miles at impact. Thus \( \delta \rho \) in Eq. (13) ranges over \( \pm 100/4050 = \pm 0.025 \). Therefore we will neglect \( \delta \rho \) everywhere in Eqs. (16) and (17) as being small compared with unity except in the term \( (1 + \delta \rho)^{-1} - u_{ho}^{-1} \), where we approximate \( (1 + \delta \rho)^{-1} \) by \( 1 - \delta \rho \), and the term \( u_{ho}^{-1} (1 + \delta \rho)^{-1} \) which we also approximate as \( u_{ho}^{-1} (1 - \delta \rho) \). Then Eqs. (16) and (17) become

\[ w_h = \frac{t}{r_o} \left[ 1 - \delta \rho - u_h^2 + \frac{Z_h}{mg_o} \right] \, d\tau + \frac{w_h}{t=0} \]

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and

\[ u_h = \int_0^t \frac{X}{mg_o} \frac{d\tau}{T} + \rho_o u_h^{\prime} - (\rho_o u_h^{\prime})^2 \delta \rho \]  \hspace{1cm} (20)

In addition, we note that

\[ \frac{d(\delta \rho)}{d(t/T)} = -w_h \]  \hspace{1cm} (21)

For \( r_0 = 4050 \) statute miles the time constant \( T = 837 \) seconds.

Let us assume that we desire to measure the position of the warhead in the plane of the trajectory using dimensionless rectilinear coordinates \( x \) and \( y \), as shown in Figure 3. If 1000 statute miles is approximately the maximum value for horizontal target distance \( r_x \), then the maximum value of \( \theta \) in the figure is approximately 0.25 radians or 14 degrees, in which case small-angle geometry can be used to compute \( x \) and \( y \). Thus if we integrate local horizontal velocity \( U_h \) directly, we obtain, approximately, the distance coordinate \( x_{r_0} \). Therefore, we can write

\[ x_{r_0} - x = \frac{U_{h_0}}{r_0} \int_0^t u_h d\tau \]

\[ = \sqrt{\frac{g_o}{r_0}} \int_0^t u_h d\tau = \int_0^t \frac{u_h}{T} d\tau \]  \hspace{1cm} (22)

where \( x_{r_0} \) is the initial \( x \).

From Figure 3 it is apparent that

\[ r_{r_0} = r \cos \theta - R \approx r(1 - \frac{\theta^2}{2}) - R \]

\[ \approx r - r_0 - \frac{\theta^2}{2} \]  \hspace{1cm} (23)

Clearly \( \theta \approx x \), and Eq. (23) becomes

\[ y \approx \frac{r_0 - R}{r_0} - \frac{x^2}{2} \]  \hspace{1cm} (24)

From Eqs. (22) and (24) we can compute the dimensionless coordinates \( x \) and \( y \), having solved Eqs. (16) and (17) for the dimensionless velocities \( u_h \) and \( w_h \), and the dimensionless radial perturbation \( \delta \rho \). To define completely the equations to be solved with the analog computer we need only rewrite Eqs. (8) and (9) for the horizontal and vertical aerodynamic forces in terms of the dimensionless variables. It is convenient for scaling purposes to compute in each case the logarithm of the horizontal and vertical aerodynamic acceleration in units of \( g_o \). Thus we obtain

\[ \log \left( \frac{X}{mg_o} \right) = B - 0.434 \frac{r}{h_o} \delta \rho \]

\[ + \log u_h + \log v_a \]  \hspace{1cm} (25)

\[ \log \left( \frac{Z}{mg_o} \right) = B - 0.434 \frac{r}{h_o} \delta \rho \]

\[ + \log w_h + \log v_a \]  \hspace{1cm} (26)

where

\[ B = \log \left( \frac{1}{2} \rho_o e^{\frac{D^2}{2}} \right) \]

\[ C = \frac{D^2}{2} \]  \hspace{1cm} (27)

and is a constant if \( C \) is a constant, or at worst a simple function of Mach number \( M \) for a Mach-dependent drag coefficient.

We have now developed the equations for obtaining the \( x \) and \( y \) coordinates of a warhead or decoy in the plane of the trajectory, and the equations are scaled conveniently for computer solution. We will see later that displacement at right angles to the plane of the nominal trajectory for decoys with initial side-velocity components is directly proportional to horizontal-distance traveled, \( x_{r_0} - x \), and can be so computed.

Analog Computer Circuit for Obtaining Trajectories in the Plane of the Motion

The analog computer circuit for solving the two-dimensional equations of motion given at the end of the previous section is shown in Figure 4. The circuit has been scaled with unity equal to 100 volts, the computer reference, and uses the dimensionless velocity and displacement variables presented earlier. The circuit as shown allows horizontal velocities up to 25,600 ft./sec., altitudes up to 200 statute miles, initial horizontal down range distance from impact of 1000 statute miles, and maximum vertical and horizontal acceleration components of 50 g's. The time scale of this particular circuit is approximately 42 times real time (\( T = 20 \) seconds instead of 837 seconds, as given in Eq. (18) for a mean reference altitude of 100 miles). This was chosen to allow convenient recording of solutions. Extension of the circuit to real-time operation involves reduction of the gain of each integrator by a factor of 42, which because of our previous experience on real-time analog simulation of orbital and reentry vehicles is known to be quite feasible.

Note that all nonlinear operations in the cir-
circuit of Figure 4 are performed using solid-state components, quarter-square multiplier elements for squaring and square-root operations, and logarithm generators for computing aerodynamic acceleration components in accordance with Eqs. (25) and (26). The logarithm generators allow particularly favorable scaling to be employed in what is normally a very difficult problem for analog mechanization. The diodes around amplifiers B6 and C6 limit the amplifier outputs in the negative voltage direction to about -0.7 volts. This prevents amplifier saturation whenever the log generators go outside their useful range (in this case approximately 2 1/2 decades). Thus aerodynamic accelerations can be computed with reasonable accuracy from 50 \(eg\) down to 0.2 \(eg\). The diode circuits around amplifiers C1 and C2 prevent the vertical and horizontal drag accelerations from taking on negative values. Thus as a reentry simulation takes place the drag acceleration is held to zero within the offset of amplifiers C1 and C2 (equivalent to less than \(\pm 2.5 \times 10^{-6} \text{eg}\)) at high altitudes. As soon as the more dense atmosphere is encountered, the drag acceleration increases smoothly from zero, with the computation accurate to the order of several percent of the value itself for 0.5 \(g\) or greater acceleration. The diodes around amplifier A1 prevent horizontal velocity \(u_i\) from going negative.

This problem was set up on an AD-1-32PB* electronic differential analyzer using quarter-square multipliers with an accuracy specification of \(\pm 0.035\) percent (\(\pm 35\) millivolts) for the squaring operation and fixed diode log generators with an accuracy specification of \(\pm 0.2\) percent over two decades of input voltage. Actual recordings of computer solutions are presented in a later section.

Note in Figure 4 that a single potentiometer, C4, sets the logarithm of \(C_{DA}/mg_0\). For a Mach-dependent drag coefficient it is only necessary to add a function of mach number \(M\) to the input of amplifier B8. Note also that single potentiometers set perturbations \(\delta u_h\) and \(\delta w_h\) in horizontal and vertical velocity.

**Generation of the Third Dimensional Coordinates for Multiple Trajectories**

In all of the discussion up to now we have assumed a two-dimensional trajectory in a nominal reference plane. If the warhead or any decol has an initial small velocity component \(v_{h_0}\) at right angles to the nominal trajectory plane, the net effect will be to yaw the trajectory plane for that particle through a small angle \(\psi = v_{h_0}/v_0\), where \(v_0\) is the initial total velocity. This will generate a dimensionless side displacement \(z = \psi(x_i - x)\), where \(x_i - x\) is the dimensionless horizontal distance traveled by the particle from the start of the reentry trajectory simulation. Using the voltage output \(4(x_i - x)\) from the circuit of Figure 4, the circuit of Figure 5 can be utilized to compute the side displacement \(z\) for any number of targets, each with arbitrary initial side velocities and hence trajectory yaw angles \(\psi\) of either polarity. Note that only \(n+1\) passive resistors are required for \(n\) targets.

**Generation of the Two-Dimensional Coordinates for Multiple Trajectories**

In the previous section we saw how the side displacement \(z\) could easily be generated for any arbitrary number of particles from the downrange distance \(x_0 - x\). Assume next that we wish to compute the trajectory variables \(x\) and \(y\) for a large number of particles, each with different initial horizontal velocity \(u_{h_0}\), vertical velocity \(w_{h_0}\), and ballistic coefficient \(C_{DA}/mg_0\). As a specific example suppose we choose for a nominal trajectory that of a particle with an initial altitude of 200 statute miles, \(u_{h_0} = 0.8\) (initial horizontal velocity of 0.8(25, 600) = 20, 500 ft. / sec.), \(w_{h_0} = 0.25\) (initial flight path angle \(\gamma = \tan^{-1}(-0.25/8) = -17.3\) degrees), and \(C_{DA}/mg_0 = 0.1\). Next assume that the maximum deviation \(\delta u_{h_0}\) in horizontal velocity corresponds to \(\pm 512\) ft. / sec. In Figure 6b are shown the analog computer trajectory solutions in this case, where the center trajectory is the nominal one, and where the trajectories on either side represent the particle path for \(\pm 512\) ft. / sec., and \(\pm 512\) ft. / sec. horizontal velocity deviations, respectively. Here the nominal trajectory impact point is approximately at the origin of the \(xy\) coordinate system of Figure 3.

Next consider the trajectories for initial vertical velocity deviations of \(\pm 512\) ft. / sec., as shown in Figure 6a. Again the center trajectory is the nominal one whereas the trajectories on either side are the perturbed ones. For orientation purposes the outline of the surface of the earth is shown at the bottom of the figure.

Finally, consider the trajectories for \(C_{DA}/mg_0 = 1\) and 0.01 in addition to the nominal \(C_{DA}/mg_0 = 0.1\). These results are also shown in Figure 6a, where the trajectories for \(C_{DA}/mg_0 = 1\) and 0.01 do not depart from the nominal trajectory until near the end of the trajectory, since this is where the large aerodynamic forces are encountered.

From the results of Figure 6 it is evident that reasonably accurate trajectories for any initial velocity components between the two limiting
extremes can be obtained by linear interpolation from the extreme trajectories. For different $CDA/mg_0$ it appears that a logarithmic interpolation is preferable. This conclusion is substantiated in Figure 7, where vertical target displacement $y$ and horizontal target displacement $x$ are recorded as a function of time for $CDA/mg_0 = 1, 0.1,$ and $0.01$. In each case points which would have been obtained for $CDA/mg_0 = 0.1$ from logarithmic interpolation between the trajectories for $CDA/mg_0 = 1$ and $0.01$ are shown. Note that they fall almost exactly on the true trajectory for $CDA = 0.1$.

Similarly, in Figures 8 and 9 are shown time-history recordings of $y$ and $x$ for horizontal velocity deviations of $\pm 512$ ft./sec. from the nominal trajectory and vertical velocity deviations of $\pm 512$ ft./sec. In each case it is evident that the nominal trajectory would have been obtained accurately by linear interpolation from the extreme trajectories.

In the previous section we established the method for generating lateral displacement $z$ for any arbitrarily large number of targets with different initial horizontal and vertical velocities, and ballistic coefficient, i.e., compute the limiting trajectories at both extremes of each initial variable and use linear or logarithmic interpolation to obtain the individual trajectories.

The above interpolation computations can be implemented with simple passive resistor networks. Assume that we have six analog computer circuits similar to that shown in Figure 4, each capable of generating a separate reentry trajectory. Let the first two circuits employ nominal initial vertical velocity $w_0$ and ballistic coefficient $CDA/mg_0$, but the extreme initial limits of initial horizontal velocity, i.e., $u_{h0} + \delta u_0$, and $u_{h0} - \delta u_0$, respectively. In each case designate the output coordinates $x_{u1}$, $y_{u1}$ and $x_{u2}$, $y_{u2}$, respectively.

Similarly, let the second pair of circuits compute the trajectory coordinates $x_{w1}$, $y_{w1}$ and $x_{w2}$, $y_{w2}$, respectively, for the extreme limits of initial vertical velocity, and the third pair of circuits compute the trajectory coordinates $x_{D1}$, $y_{D1}$ and $x_{D2}$, $y_{D2}$, respectively, for the extreme limits of $CDA/mg_0$.

Finally, assume that a seventh analog circuit generates the nominal trajectory coordinates $x_{ref}$, $y_{ref}$. Then the interpolations described previously can be implemented to compute $x(t)$ by means of the circuit shown in Figure 10, which is actually a mechanization of the formula

$$x = x_{ref} + \frac{\partial f}{\partial u_h} \Delta u_h + \frac{\partial f}{\partial w_h} \Delta w_h$$

$$+ \frac{\partial f}{\partial \log CDA} \log \frac{CDA}{mg_0}$$

This formula simply consists of the zeroth and first order terms in the Taylor series expansion for $x$ about the nominal trajectory $x_{ref}$. The difference voltages $x_{u1} - x_{ref}$ and $x_{ref} - x_{u2}$ are applied across an array of series resistors as shown at the top of Figure 10. From these resistors we can obtain voltages $x_1(t), x_2, \ldots$ representing the term $\partial f/\partial u_h \Delta u_h$ in Eq. (28) for targets number 1, 2, 3, \ldots. The distribution of series resistor values corresponds to the distribution in initial horizontal velocity perturbations for all the targets. Similarly the second and third terms on the right side of Eq. (28) are generated for different targets as voltage outputs of series resistor arrays at the middle and bottom of Figure 10. The distribution of series resistor values in the middle array corresponds to the distribution in initial vertical velocity perturbations for all the targets, while the distribution of series resistor values in the bottom array corresponds to the distribution in the logarithm of $CDA/mg_0$ for all the targets. For a given target the horizontal displacement coordinate $x$ is obtained by summing the voltages representing each of the terms in Eq. (28) as implemented with the passive-resistor network shown in Figure 10.

The circuit for obtaining vertical displacement $y$ for each of the targets is similar to that for $x$ in Figure 10, except that the $y$ voltage outputs from each trajectory simulation are used instead of the $x$ outputs. Note that the circuit of Figure 10 uses linear interpolation between the nominal trajectory and the perturbed trajectory on either side, whichever is appropriate. If less accuracy is required or if the maximum perturbations that need to be considered are small enough, the nominal or reference trajectory can be selected with the minimum initial horizontal velocity, vertical velocity, and $CDA/mg_0$ to be considered. Then all perturbations are positive and only four trajectories ($x_{ref}$, $y_{ref}$, $x_{u1}$, $y_{u1}$, $x_{u2}$, $y_{u2}$, $x_{D1}$, $y_{D1}$) need be computed instead of the seven shown in Figure 10.

In Figure 5 we showed the circuit used to generate different side-displacement $z$ for various targets using horizontal displacement $x = x$ as measured from the initial position $x$. In Figure 10 each target has a different horizontal displacement $x$, from which it is possible to generate a lateral displacement $z$ proportional to $x_0 - x$ by
a passive-resistor network which adds a voltage proportional to $x$ to a constant negative bias voltage representing $x_0$. Here both signs of initial side-velocity perturbation can be taken into account by using for the reference-trajectory the plane corresponding to the maximum negative initial side velocity.

Conclusions

It is concluded that the reentry trajectory in two dimensions can be simulated with a modest amount of analog-computing equipment over a wide range of initial conditions and ballistic coefficients. The third dimension is obtained as a coordinate displacement proportional to horizontal-distance traveled. Reasonably accurate trajectories for a large number of particles with different initial velocities and ballistic coefficients can be obtained with passive resistor circuits by linear interpolation from limiting trajectories. A minimum of four two-dimensional trajectory circuits are required for this, although the use of seven trajectory circuits improves the accuracy considerably. In either case the amount of analog equipment required is not formidable. It should be noted that if accurate simulation of target velocities or accelerations is required (particularly in the case of accelerations) the simple linear interpolation formulas proposed in this paper would result in large errors in that portion of the trajectory wherein the maximum accelerations occur. Perhaps computation of each individual two dimensional trajectory is needed in this case.
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Figure 1. Basic Geometry in the Plane of the Trajectory

Figure 2. Geometry of the Drag Forces
Figure 3. Coordinates in the Plane of the Trajectory
Figure 4. Computer Circuit for the Two-Dimensional Trajectory Equations

Figure 5. Circuit for Generating Side Displacement Z for n Targets
Figure 6a. Effect of Initial Vertical Velocity Perturbation and Different $C_D A/mg_o$ on an ICBM Trajectory

Figure 6b. Effect of Initial Horizontal Velocity Perturbation on an ICBM Trajectory

Figure 7. Effect of $C_D A/mg_o$
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Figure 8. Effect of Initial Horizontal Velocity Perturbation

Figure 9. Effect of Initial Vertical Velocity Perturbation
Figure 10. Circuit for Computing Horizontal Distance Coordinate for Multiple Trajectories