THE USE OF COMPUTERS IN ANALYSIS
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Introduction

Automatic computers, both analog and digital, have attained a prominent place in most modern engineering curricula. The computer is recognized as an important engineering design tool permitting the student to test the efficacy of a large number of design hypotheses to determine an optimum design. In these applications the computer is favored because it enables the student to dispense with laborious hand calculations and because it familiarizes the engineering student with techniques which will subsequently be valuable to him in industry. The application of automatic computers to another category of engineering courses - courses in methods of analysis - is more controversial.

Engineering analysis courses include such subjects as electric circuit analysis, dynamics, thermodynamics, fluid mechanics, and field theory. The chief objective of these courses is to instill in the student a profound understanding of fundamental engineering practices and systems and to give him a constructive insight into the response of engineering systems to various excitations. The emphasis in such courses is on generalization rather than on specific solution as in engineering design. Since automatic computers can only be used for numerical analysis - the solution of problems in which all variables and parameters are specified numerically - it has been argued that computers have little or no place in courses in engineering analysis. According to this line of argument the availability of a computer enables the student to "short circuit" the learning process. Homework and classroom problems in courses in analysis are designed ideally in such a way that by carrying out the actual computation, the student gains an insight into the physical system and obtains benefits far exceeding the interpretations that he is able to draw from the final numerical solution. Furthermore, the availability of computers discourages the student from a conscientious effort to find the general and therefore more valuable and elegant solution, rather than a numerical solution applicable only to a specific situation. It has been stated that the availability of analog computers in the late 1940's and early 1950's greatly slowed the orderly development of the field of nonlinear control system theory. Most engineers found it much easier to solve a specific problem on a computer rather than to seek and develop a general theory for handling nonlinear system analyses.

At the other extreme are educators who are so computer-oriented that they recommend that computers be utilized in every course in engineering analysis at least to some extent. They argue that although, as yet, the generalizations which can be drawn from numerical analysis are limited, this is only a reflection of the relative infancy of the computer field. They feel that soon a new generation of automatic digital computers will be available. These new computers will make it possible for an engineering analyst or scientist to "converse" with the automatic brain, that is, to use the computer as an extension of his own thinking processes. In this way the use of computers will permit the drawing of generalizations beyond the scope of the human mind. Accordingly, it is argued that in order to prepare the engineering educational field for this day it is necessary to begin now to reorient thinking in the direction of computers and computing techniques.

It is not the authors' intent in this paper to attempt to resolve these two lines of argument. Along with most other engineering educators we recognize the importance of mathematical sophistication and the fostering of the students' interest and ability to make significant generalizations. At the same time we realize that the computer age is upon us and that no area of technological and scientific endeavor can completely escape its impact. It is the objective of this paper to demonstrate by means of a number of specific examples how the utilization of computers can be compatible with the basic objectives of courses in engineering analysis. Foremost among these examples are two categories of computer utilization:

1. The application of computers to aid the student in the visualization of dynamic or mathematical phenomena.
2. The opening up of new approaches to the explanation of system behavior - approaches which are out of reach of conventional analytical methods.

Monte Carlo Methods and Random Walk Phenomena

The analysis of fields as they occur in such engineering disciplines as electrical circuit analysis, heat transfer, and fluid mechanics generally follows a deductive approach. By applying basic physical laws such as the law of conservation of mass and energy and the principle of continuity to a small bounded region of the field (the differential element) the basic partial differential equations characterizing field problems are derived. These
equations include, among others, Laplace's equation, Poisson's equation and the wave equation. Mathematical techniques such as separation of variables, conformal transformations, etc. are then applied to adapt the general equations to the two specific boundary and/or initial conditions. This finally results in two mathematical expressions characterizing the equipotential and stream lines within the field and specifying transient variation of the field potential at specific points. Although this approach is of unquestioned value and utility both to physicists and engineers, many students will find an additional alternative approach both interesting and stimulating.

The second approach is governed by the recognition that the uncertainty principle applies to any specific particle under the influence of the potential field. In other words, the classical equations apply only to the statistical average of a fluid particle in a fluid field, electron in electrodynamics, etc.) will not follow the stream line which constitutes the solution to the classical problem, but will rather describe a "random walk". Accordingly, to solve a field problem, i.e., determine the potential at some point within the field, by this approach it is necessary to perform a series of consecutive random walks all commencing from the same point and to tabulate the results of these walks.

While this technique will rarely be able to compete with the more conventional techniques in terms of efficiency, an important educational advantage may be gained in familiarizing the student with the field process as it actually occurs in nature. By permitting the student to compare the results of classical field problem solutions with random walk solutions he will gain a deeper insight into the significance of the mathematics, that is, generalizations which he draws from a mathematical analysis will be more meaningful.

A practical method for performing such random walk solutions, originally presented by Chuang, Randa, and Winterkorn will now be described. Analog computing techniques are employed since it is desired to present the data visually and to facilitate manipulation of system parameters.

A technique which is well-established in digital computation involves the use of a sequence of random numbers as computer inputs. By making a sufficiently large number of computer runs, and by taking a weighted average of the results of these runs, convergence to the desired solution can be obtained. This technique can be applied in analog computation by exciting the computer system with a random noise generator - a voltage generator whose output amplitude is governed by a stochastic expression. The repetitive mode of analog computer operation is useful in performing such calculations because a large number of separate runs, each with a random noise input, can be completed within a reasonably short time.

The boundary value problem for which the Monte Carlo method is applicable belongs to a family of generalized Dirichlet problems of the form

\[ D_1 \frac{\partial^2 \phi}{\partial x_1^2} + D_2 \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial \phi}{\partial x_1} K_1 + \frac{\partial \phi}{\partial x_2} K_2 = 0 \]

where \( K_1 \) and \( K_2 \) are arbitrary functions of the independent variables \( x_1 \) and \( x_2 \), respectively, while \( D_1 \) and \( D_2 \) are constants. The boundary \( C \) is an arbitrary finite closed curve - a Jordan curve. The Monte Carlo method provides a solution in the form of the field potential existing at some point within the field. It does not provide equipotential curves as do the more conventional analytical and analog simulation techniques.

In principle the Monte Carlo method involves selecting a point of interest within the field and from this point commencing a random walk. That is, a sequence of small steps are taken such that each step begins at the end of the preceding step but proceeds in a random direction but always parallel to the two perpendicular coordinate directions. Eventually each such random walk will reach a point on the field boundary \( C \). The potential at this boundary point is recorded and a new random walk is commenced. Provided that enough random walks are taken, and provided that each step in the walk is sufficiently small, it can be shown that the weighted average of the boundary intersections will converge to the potential existing at the point in question within the field.

Referring to Figure 1, assume that the solution of Laplace's equation

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]

is required at some point \( P \) inside a region bounded by \( C \) upon which the values of \( \phi \) are prescribed. Starting at \( P \) a random walk is taken in which steps of equal length are made in either a positive or negative direction parallel to the \( x \)- or \( y \)-axis. The walk continues until it reaches \( C \) when it is terminated and the value of \( \phi \) on \( C \) say, is recorded. This process is repeated \( n \) times and it may be shown that

\[ \phi(P) = \lim_{n \to \infty} \sum_{i=1}^{n} \phi_{C_i} / n \]

In the computer method to be described, the random walk is actually performed by the beam of
A cathode-ray oscilloscope. When this beam contacts the field boundary as defined by a mask placed over the face of the oscilloscope, the random walk is terminated and a new one begins.

Some auxiliary circuitry is required to cause the oscilloscope beam to describe the random walk. It has been shown that when an electrical circuit is subjected to an exciting input voltage of Gaussian white noise, the response current of the circuit is a Markov stochastic process and that consequently, the conditional probability density function of the current satisfies the equations governing random walk phenomena. For the solution of two-dimensional generalized Dirichlet problems two independent resistance-inductance circuits with nonlinear resistance are required. The equations describing the response of these circuits subjected to two independent Gaussian white noise sources \( F_1(t) \) and \( F_2(t) \) are

\[
\begin{align*}
\frac{dy_1}{dt} + K_1 &= F_1(t) \\
\frac{dy_2}{dt} + K_2 &= F_2(t)
\end{align*}
\]  

where \( K_1 \) and \( K_2 \) are functions of \( y_1 \) and \( y_2 \). The voltages generated by solving Equations (4) on the computer are employed to drive the oscilloscope beam in the \( x \) and \( y \) directions. Accordingly, the solution of a Dirichlet problem of the type described by Equation (1) takes the following steps:

1. Two resistance-inductance circuits governed by Equation (5) are simulated. In general these governing equations will be nonlinear.
2. Two independent voltage sources of white noise \( F_1 \) and \( F_2 \) with adjustable spectral-density \( D_1 \) and \( D_2 \) respectively, are employed to drive the two resistance-inductance circuits.
3. The voltage outputs \( y_1 \) and \( y_2 \) of the two nonlinear circuits are used to drive the horizontal and vertical inputs respectively of a cathode-ray oscilloscope. The prescribed boundary \( C \) is introduced in such a way as to make detectable the impingement of the oscilloscope beam on the boundary.
4. A value of \( F(Q) \) must be generated to correspond to any point \( Q \) on \( C \) on which the beam makes contact with the boundary.
5. A continuous record of successive generations of \( F(Q) \) must be kept or, alternatively, successive values of \( F(Q) \) must be continuously recorded. In either case the mean value of all the individual results must be determined.
6. At each occurrence of incidence, the oscilloscope beam must be reset to the initial point inside the boundary \( C \), the point at which the solution of Equation (5) is desired and at which each random walk of the oscilloscope beam must begin.

This cycle of operation must be repeated continuously until such time as the value of the mean of the summed boundary values \( F(Q) \) ceases to change. A block diagram and some physical detail of a computer system for the solution of Laplace's equation

\[
\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0
\]

is shown in Figure 2. In this case the potential distribution along the boundary was such that \( C \) could be divided into two parts by a straight line so that the arc length (which is a function of \( x_1 \) and \( x_2 \)) of each part was a single-valued function of the length of the dividing line. In field problems of heuristic interest it is nearly always possible to find such a dividing line. Under these conditions two diode function generators suffice to obtain the value \( F(Q) \). The incidence of the beam on either of the two sections of the boundary is detected by means of one of two phototubes which trigger a gate. This in turn permits the output of the two diode function generators to apply voltages determined by the boundary point coordinates to the \( F(Q) \) register. By suitable timing circuits, each run can be made to last from one to 3 seconds. In this manner a reasonably large number of runs can be completed in a very few minutes; at the same time the trajectory of the oscilloscope beam during each run will be clearly visible to the observer. The opaque mask shown in the figure is opaque only to the detecting photocells. It transmits sufficient light to enable the observer to see the movement of the oscilloscope beam quite clearly.

The generation of the variables \( y_1 \) and \( y_2 \) in Equation (4) is effected by means of a circuit of the type shown in Figure 2. This circuit requires two analog summers, an analog integrator and a function generator. Low frequency noise sources of Gaussian amplitude distributions having the necessary power spectra are commercially available. Such devices generally employ a gas tube such as a thyratron as the basic noise source.

Experiments have shown that good convergence of the random walk solutions to the analytical solutions in one- and two-dimensional boundary value problems are obtained if approximately 300 random walks are carried out.

Other Analytical Techniques

There are a number of other ways in which computers may be used effectively to bring a deeper understanding of some phase of analysis.

In finding the complex roots of a polynomial equation, for example, the method of steepest descent provides a useful technique. In this method a related function is minimised in a continuous and automatic manner. If the function
has a minimum at only those points where the polynomial equation has roots, then finding the minimum is equivalent to locating a root.

A description of the automatic nulling process associated with the method of steepest descent on an analog computer has been recently described. A simple variation of this method has even wider application. Once the computer has obtained the root of the polynomial equation by the nulling process it may be operated in an automatic tracking mode. Thus, if one or several of the coefficients of the polynomial are varied, the computer will follow the root to its new location. This technique is extremely useful to control-system engineers who use the root-locus tracking to obtain information about the stability of control systems.

Another analytical approach which can be clarified for the student by the use of computers is conformal mapping. Any attempt to map the z-plane into the w-plane by means of the function \( w = f(z) \) must necessarily be done point by point. With an analog computer, however, a curve in the z-plane can be continuously mapped into the w-plane in a matter of seconds. Tomlinson has described the circuitry required to map a circle into an airfoil shape.

Both the Fourier transform and the inverse Fourier transform of a function can be obtained on an analog computer. A by-product of this method are the Fourier coefficients in the expansion of the function. Thus, Fourier analysis can be done on either a digital or an analog computer. Again, the advantage of the latter is in helping to visualize the process and making it easy to see the result of changes in the function being analyzed.

Finally, mention should be made of the work that Bellman has done in dynamic programming. This technique was developed to solve variational problems which arise in economic, industrial, engineering, and biological contexts and for which the classical calculus of variations methods are inadequate. By utilizing the capabilities of the digital computer Bellman has developed a new technique which has the advantage that it handles linearities or nonlinearities, constraints or not, implicit or explicit functionals, in precisely the same manner, and with the same basic equation.\footnote{Nonlinear Analysis}

In this "space age" more attention is being given to nonlinear analysis. As it becomes increasingly important for mathematical models to match physical situations more closely, more thought must be given to topics like nonlinear differential equations, both ordinary and partial. As we dispense with incompressible fluids, perfect gases, isotropic solids, frictionless materials, weightless bars, etc. we are forced into the realm of the nonlinear. Unfortunately this is unexplored territory because the analytical techniques presently in use are linear in nature. This is true of such giants in analysis as the Laplace transform, the Fourier transform, the Cauchy residue theorem and others. Attempts to extend these tools to nonlinear problems have met with little success.

Thus, for lack of adequate analytical techniques for solving nonlinear problems, the use of computers becomes mandatory. The analog computer, in particular, is useful for studying nonlinear problems. Such studies can help the student visualize nonlinear system behavior and may even lead to important generalizations and new analytical methods. Certain modifications of the analog computer are necessary, however, in order to adapt it to nonlinear studies. These consist of repetitive operation, automatic programming, and means for changing the degree of nonlinearity.

Repetitive operation is well known by now as most modern computers have provisions for it. Older installations can be easily converted by the addition of an integrator whose output ramp is used to energize relays which in turn clamp the integrators in the problem. The repetition rate is controlled by the rate at which the ramp is generated. Care must be exercised so that the repetition rate is not so high that the problem frequencies exceed the frequency-response limitations of the amplifiers.

Automatic programming consists of changing problem parameters by predetermined increments. This is done by generating a monotonically increasing step function. The circuit for this is energized by the repetitive reset pulses and is shown in Figure 3. The step function generated can be used in a number of ways in parameter studies.

One of the most common changes that has to be made in a problem is changing a potentiometer. This can be done by applying the output of the step-function circuit to a multiplier which then multiplies a variable by a constant. Figure 4 shows the circuit for accomplishing this.

Automatic changes in initial conditions can produce a family of solutions in a short time. A typical example of this is shown in Figure 5.

The analog computer is basically a device for solving linear problems. The introduction of nonlinearities increases the complexity of the equipment which often results in decreased reliability. It is now possible, however, to represent a wide range of nonlinearities in a simple manner. This is done by using Quadratrons, nonlinear components which supply the square of an input voltage but which can provide other nonlinearities by simple changes in the auxiliary circuitry. Figure 6 shows the basic circuit with the Quadratron denoted by QP.
The characteristic of the Quadratron is a result of the silicon carbide varistor it contains. In this type of varistor the current varies as some power of the voltage. By combining the varistor with ordinary linear resistance it is possible to make this power very nearly two. Current is ideally supplied to the Quadratron by an operational amplifier. Thus the Quadratron-amplifier combination of Figure 6 forms the basic circuit.

The basic circuit is useful, for example, in representing square-law damping in a mechanical system. Note especially that the symmetry about the origin of the circuit's transfer function is exactly what is required in this type of simulation. There has been a tendency in the past on the part of some educators to feel that they have reached a high level of sophistication when they introduce their classes to second-order differential equations containing square-law damping. We can, however, study a large number of systems each having a different degree of damping quite easily.

Consider the nonlinear differential equation

\[ m \ddot{x} + \mu \dot{x}^n \text{sgn} \dot{x} + k x = f(t) \]

\[ x(0) = \dot{x}(0) = 0 \] (6)

The signum function, \( \text{sgn} \dot{x} \), is defined by

\[ \text{sgn} \dot{x} = \begin{cases} 
-1 & \text{if } \dot{x} < 0 \\
0 & \text{if } \dot{x} = 0 \\
+1 & \text{if } \dot{x} > 0 
\end{cases} \] (7)

Figure 7 shows the circuit for obtaining \( \dot{x}^n \text{sgn} \dot{x} \) for \( \frac{1}{2} \leq n \leq 2 \). The change in the exponent \( n \) is accomplished by means of the potentiometer in the circuit. Solutions to Equation (6) are obtained by combining a number of auxiliary circuits as shown in Figure 6.

Some of the advantages of studying a nonlinear differential equation in this manner are as follows:

1. The student is made to realize that the linear case is a special case — it is one case out of an infinite number of nonlinear cases. This helps to present linear and nonlinear problems in the proper perspective.
2. The student obtains an idea of how much the exponent can differ from unity before the results change appreciably from the linear case. This gives him a feel for the amount of linearization that can be applied to a given problem.
3. The student can examine the family of output curves and relate the changes to the changes in the exponent. He may thus be able to generalize the results.
4. The student becomes familiar with nonlinear systems in general through the visual presentation from the analog computer. This point is discussed later in connection with limit cycles.

More complex equations may be studied by an extension of the methods described. In this way greater insight can be gained into the nonlinear realm which so far has not yielded to general analytical techniques.

A classical nonlinear differential equation that has been widely studied is the van der Pol equation, usually written

\[ \frac{d^2x}{dt^2} - \epsilon (1-x^2) \frac{dx}{dt} + x = 0 \] (8)

This equation describes an oscillatory system having variable damping. The damping depends not only on the displacement but also on the parameter \( \epsilon \). Different types of solutions are obtained depending on the magnitude of \( \epsilon \) in comparison with unity. It is a simple matter to solve Equation (8) on an analog computer with the techniques already described.

One of the most important features of the study of nonlinear differential equations on an analog computer is the insight it gives the student into limit cycles. These are closed curves representing a steady-state periodic oscillation determined only by the characteristics of the equation itself, and independent of the way the oscillation is started. If the initial point is inside the limit cycle, the solution curve will spiral outward until it reaches the limit cycle. If the initial point is outside, the solution curve spirals inward. It would be too time-consuming to see this phenomena if the solution curve had to be obtained and plotted point by point.

Network Topology

Since 1954 much work has been done in the study of electrical networks from a topological viewpoint. Any network diagram represents a steady-state periodic oscillation determined only by the characteristics of the equation itself, and independent of the way the oscillation is started. If the initial point is inside the limit cycle, the solution curve will spiral outward until it reaches the limit cycle. If the initial point is outside, the solution curve spirals inward. It would be too time-consuming to see this phenomena if the solution curve had to be obtained and plotted point by point.

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With the advent of parametric amplifiers there has been a renewed interest in the analysis of networks containing periodic parameters. When even one periodic circuit element is introduced in a lumped, linear, finite, passive, bilateral, time-invariant network, the analysis of the resulting network becomes a formidable task. Leon and Bean\(^2\) have shown how a digital computer may be useful in this case.

**Conclusion**

Automatic computers have a definite place in engineering analysis. Analog computers, in particular, are useful in helping the engineer visualize dynamic phenomena and in giving him a feel for the effect that various parameters have on the solution. The use of automatic programming techniques are helpful in making generalizations which may be impossible to achieve without the time-saving feature of the computer.

While the use of computers in design may be important to the engineer (especially from an economic standpoint), the real contribution of the computer will come from its use in analysis. With the assistance provided by the computer the analyst can make significant progress in extending man’s knowledge of his environment.

**References**


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1. A random walk in the Monte Carlo method
2. Solving Laplace's equation by the Monte Carlo method (after Chuang, et al)
   a) Block diagram of system
   b) Circuit for generating random walks
3. Step-function generator

4. Automatic changing of a constant coefficient
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5. Automatic changing of initial conditions and the resulting solutions

6. Basic circuit and transfer function of the Quadratron

\[ e_o = -0.01 e_l^2 \text{ (sgn } e_l) \]
7. Circuit for obtaining an arbitrary exponent

8. Block diagram of system for solving a nonlinear differential equation