ALGY - AN ALGEBRAIC MANIPULATION PROGRAM

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Summary

In a great variety of scientific problems, the reduction of complex analytical expressions is highly desirable but often such reductions, although straightforward, are extremely lengthy and laborious. The following paper describes a program written for a high-speed digital computer which accepts algebraic expressions as input and outputs a similar set of modified expressions. The description includes the definition of the ALGY operations used to obtain the desired results, followed by an explanation of the logical flow of the program, and concluding with a description of future operations which will be incorporated into the system.

Introduction

The kinds of problems which initiated interest in general purpose high-speed digital computers were, for the most part, problems which involved an extreme amount of arithmetic. Without computers, in many cases, no attempt could be made to solve them, as valuable as their solutions were deemed to be, not only because of the time required to perform the computations, but also because of the very small probability that the results after months of hand calculation would be correct. With the advent of the electronic computer, the arithmetic involved in these problems became a trivial matter and solutions were easily effected with a high degree of reliability.

Recently, an analogous difficulty has arisen in solving another kind of problem. We wish to solve systems of differential equations by perturbation methods. These problems, rather than involving arithmetic, required an overwhelming amount of algebraic manipulation. The only feasible way to handle this kind of problem is, again, to "let the computer do it", and it was for this purpose that the computer program called "ALGY" was developed.

ALGY is an interpretive routine through the use of which the programmer or mathematician may instruct the computer to perform certain algebraic manipulations. ALGY is basically an elaborate scheme for the manipulation of alphanumeric material. The input and output used with ALGY are alphanumeric algebraic statements. In this paper, we will give the definitions pertinent to the ALGY program, describe the currently available algebraic manipulations in a brief description of the flow of the ALGY program, demonstrate by use of an example an ALGY program, and finally, give a brief outline of future algebraic operations and applications.

Definitions

In order to understand the manipulation which ALGY performs, it is necessary to define a few special terms. The basic building block in ALGY is the BCD character. These characters are combined in various ways into quantities. There are three types of quantities in an ALGY program: numeric, such as 51/725; special, such as plus, minus, period, parentheses, dollar sign, asterisk and quasi-alphabetic, such as cos 5X. Because it is desirable in algebraic manipulation to do exact arithmetic, all numbers are represented as fractions. The restriction on the numeric quantity is that the total number of decimal digits of the numerator and denominator must be less than or equal to 15. Under this restriction, arithmetic operations are exact. The quasi-alphabetic quantities, also, must have less than 16 characters in them and they must begin with a letter. Those restrictions are mechanical ones and could be removed by going to multiple arithmetic precision and using larger storages. Of course, in both numeric and quasi-alphabetic quantities, no special symbols can appear.

A group is an algebraic expression contained within a plus, minus or a parenthesis. It may contain several quantities, for example

-13/294X52*sin2X*Coef

is a group.

An expression is any algebraic combination of groups and/or quantities which are terminated by a period. Expressions are tagged with a name. This name may also be a quantity in another expression.

Throughout the remainder of this paper, reference will be made to different programs, the ALGY system and particular ALGY programs. The mathematician writes the ALGY program which is processed by the ALGY system. ALGY does not manipulate equations, but only expressions. However, it is easily seen that for algebraic manipulations, expression manipulation is sufficient.

ALGY Operations

There are a few general restrictions on the size and type of algebraic expression that ALGY can handle. Any expression can contain at most 4500 BCD characters. Also, all exponents must be positive integers and the only available symbols are the usual 64 BCD characters.

The currently available basic ALGY commands are EQAT, INQT, BUGG, OPEN, SBST, FCTR, TRGA and DORR. When the ALGY system was being designed, we felt that these commands constituted a minimal system to perform the algebraic manipulations that we were interested in performing. Following
is a brief description of each command. It is very easy to learn to use ALGY. Two hours of instruction are all that is usually required.

**EOAT:**
Equate merely records on tape the left and right hand sides of the equation. The algebraic expression on the right hand side of the equate symbol is always preceded by its name, that is, the symbol used on the left of the equate symbol. This method uniquely determines all expression recorded on the tape.

**INQT:**
Internal equate renames an expression already on tape, preserving the recorded name. This allows an algebraic expression to be re-eqauted without having to write the whole expression over again.

**BUGG:**
Bugg is essentially an "unequate" operation. It searches the tape for the tag that is to be "bugged" and deletes it. This is useful if the user wishes to define a particular variable several different ways during a single ALGY program.

**OPEN:**
Open removes parentheses from an algebraic expression. To do this, it performs all algebraic multiplication necessary, grouping identical terms, and sorting in a quasi-alphabetical manner.

**SBST:**
SBST substitutes one or more expression in a given expression. The routine inserts parentheses about each expression substituted.

**FCTR:**
FCTR factors a given expression with respect to a single variable, which may be exponentiated with the option of equating its coefficient to a given symbol for future reference. The expression may be factored with respect to several variables with the restriction that if a particular group contains two or more variables to be factored, it can be factored with respect to only one of the variables.

**TRGA:**
Trig A expands a product of sin and cos functions of a given argument to a sum of sin and cos functions of multiple angles. An exponentiated sin or cos function would fall under this category.

**DONE:**
Done is a control word which allows several independent problems to be processed during the same run.

**Logical Flow of ALGY**
ALGY accepts algebraic expressions as input, processes these expressions in the manner in which the user has programmed, and outputs a similar set of expressions. To accomplish this, a logical sequence of events is followed.

The algebraic equations are coded in the ALGY format which in essence is simply rewriting them on coding sheets, using English letters instead of Greek symbols when applicable, followed by all the ALGY operations necessary to obtain the desired results. After the program is key punched it is submitted for recording on tape.

ALGY accepts the coded statements, printing and performing all operations directed by the input program. A very simple example follows to facilitate explaining the logical flow of ALGY. Consider

\[
e(f,g) = (fg + 1)^3
\]

\[
f(x) = 1 + Ax + 1/2 A^2 x^2 + 1/6 A^3 x^3
\]

\[
g(x) = Bx - 1/6 B^3 x^3 + 1/120 B^5 x^5 - 1/5040 B^7 x^7
\]

and suppose the factors of \(x^{11}, x^9, x^7, x^4\) and \(x^3\) of the function \(e\) are desired, neglecting all terms of order greater than 11. The ALGY input program would appear as follows:

**EOAT E** = \((FG + 1/1)^3\) \(\text{eq} (\text{Equate E} = (FG + 1)^3)\).

**EOAT F** = \(1/1 + A^1 X + 1/2 A^2 X^2 + 1/6 A^3 X^3\).

**EOAT G** = \(B^0 X - 1/6 A^3 X^3 + 1/120 A^5 X^5 - 1/5040 A^7 X^7\).

**SBST E/F,G**. \(\text{eq} (\text{Substitute in E, the expressions equal to F and G})\).

**OPEN E**. \(\text{ie} (\text{Remove all parentheses})\).

**FCTR E/X^1,X^11,X^9,X^7,X^4,X^3,X^2,X^1,E_11/E_9,E_7/E_4,E_3/E_2,E_1/E_0,ER**. \(\text{ie} (\text{Factor E with respect to X}^1, \text{call the coefficient E}_1, \text{factor the remainder of E with respect to X}^9, \text{calling its coefficient E}_9, \text{etc., tagging the remainder of E after all factoring is completed, ER})\)

**EOAT X = 0/1.**

**SBST E11/X.**

**OPEN E11.**

**DONE.**

ALGY will accept the first three **EOAT** commands, print and store the algebraic expressions on tape, where each expression is identified by its tag on the left-hand side of the equate symbol. It then accepts the **SBST** command, searches the tape for the expression equal to \(E\) and reads it into memory. The routine locates and reads the expression equal to \(F\), and examines \(E\), substituting the \(F\) expression with parentheses around it each time it appears in \(E\). After \(F\) has been substituted, the same procedure is done for the expression \(G\). The resulting substitution in \(E\) would appear in print as follows:
E = ((1/1 + A*X + 1/2*A^2*X^2 + 1/6*A^3*X^3)*(B*X - 1/6*B^3*X^3 + 1/120*B^5*X^5) - 1/5000*B^7*X^7) + 1/1)

ALGY accepts the OPEN command and commences removing the parentheses in the E expression above. The results are printed and stored on tape. The system then enters FACTOR which will factor E with respect to each variable requested. It then prints and stores the coefficients of each factor as well as the new factored E expression as follows:

\[ E_{11} = E_{11}(X^n, A, B) \text{ where } n \text{ assumes all integers 1 through 19} \]
\[ E_9 = E_9(X, A, B) \]
\[ E_7 = E_7(X, A, B) \]
\[ E_4 = E_4(X, X^2, A, B) \]
\[ E_3 = E_3(A, B) \]
\[ E_2 = E_2(X, X^2, A, B) \]
\[ E = E_{11}*X^{11} + E_9*X^9 + E_7*X^7 + E_4*X^4 + E_3*X^3 + E_2 \]

To eliminate all higher order terms in \( E_{11} \), simply EQAT x to zero, SBST x into \( E_{11} \) and then, \( E_{11} = E_{11}(A, B) \).

In this manner the coefficients of \( X^{11} \) and \( X^3 \) can be accurately determined with a minimum of effort on the part of the user. The computer cost of the solution for a few typical problems is less than 1/6 the cost of the solution obtained by manual labor, assuming that the man performing the algebraic manipulations is as reliable as the computer.

Below is a flow diagram for the ALGY system. Because of the logical complexity of each subroutine, detailed flow charts have not been included in this paper.

Future Operations

We will tremendously increase, during the next few years, the scope and number of allowable ALGY commands. Because of the specific problem in perturbation theory for which ALGY was originally designed, we will immediately develop two more trigonometric manipulations called TRIG B and TRIG C. TRIG B allows angle variables to be separated using the laws of addition for the sin and cos functions. TRIG C is, in a certain sense, the inverse operation to TRIG A. In TRIG C, sins and cos of multiple angles are reduced to powers of the sin and cos of the angle.

There are many algebraic operations that suggest themselves now that we have the basic tools for algebraic manipulation developed. Our future plans include differentiation, restrictive forms of integration and solutions of linear systems using determinants. The restriction on the exponents will be lessened. We have already found several cases where "super" ALGY commands would be of value. For instance, it would be extremely desirable to have a command to generate a polynomial of given degree when only the ith term is given. There are also numerous form of factorization which suggest themselves. It should also be noted that the ALGY coded program is a straight-flow program just as the first numerical programs were. Undoubtedly, we will develop a loop technique for the ALGY system.

Conclusion

ALGY represents a first step in a new form of computer usage. ALGY is not a general problem solver. If the mathematician does not know how to algebraically manipulate his equations, ALGY can be of little help. But ALGY is an extremely powerful tool in the hands of an intelligent user. It enables the mathematician to consider and to solve problems that he would otherwise never consider because of the large amounts of algebraic manipulation necessary for a solution. It enables him to try different forms of a solution and use different approaches to the same problem, where before he was often committed to just one approach because of the large amount of time necessary to verify that one method.