On Prediction of System Performance from Information on Component Performance

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INTRODUCTION

THE PURPOSE of this paper is to propose some building blocks for a systematic approach to prediction of the performance or reliability of complex equipment from information on component performance. Particular attention is given to the use of information on ways in which components are believed or known to be interdependent, functionally as well as structurally.

The reliability of a system is defined as the probability \( P \) that its satisfactory operating life under stated conditions is not less than a specified time \( T \). This probability can, in principle, be estimated directly from life tests of a large number of systems. In practice, however, this is usually undesirable or impossible. In particular, it would often be desirable to make at least approximate estimates of \( P \) before a complete system has actually been assembled.

This paper presents a systematic approach for examining engineering design information, data on component performance, and information on conditions of use, in order that statistical techniques may be used to obtain an estimate of \( P \).

It is assumed that there may be obtained a collection of statements such as the following, describing the conditions on component behavior which permit satisfactory operation of the system:

"The system will operate only if properties \( X_B \) and \( Y_C \) of components \( B \) and \( C \) satisfy a specified relation (such as \( X_B + Y_C \geq \text{constant} \))."

"The system will operate only if at least one of components \( D \) and \( E \) has not failed."

It is assumed that this collection of statements contains all the essential relations determining the dependence of system performance on component performance. Then \( P \) is equal to the probability that all of these conditions will be met for the length of time \( T \).

Additional information of the following kind may be used:

"Components \( A, B, C \) will all fail at once if certain unusually extreme atmospheric conditions prevail."

"As the performance of component \( A \) deteriorates, the load on component \( B \) is decreased and deterioration of component \( B \) is slowed down."

These statements illustrate interrelations among components in their response to the conditions under which the system is operated. Information of this kind is discussed below in the section on interdependence of component failures.

It is shown in some simple hypothetical examples how this additional information on interdependence among components may be introduced into a mathematical expression for \( P \).

In the context of this approach to the analysis of system performance, the following problems are considered: 1) estimating and giving confidence intervals

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for $P$ from data obtained in tests of components and subassemblies; 2) using the mathematical expression for $P$, together with information about costs, to aid in making decisions about design changes such as the addition of redundant elements, and 3) using the mathematical expression for $P$ to simulate the effect of component specifications on system performance.

Preliminary Remarks; The Problem of Detail

It is important to state what sort of composite entity is considered in this paper to be a "system," and to indicate how the "components" of a system are viewed. A system is understood to be any equipment, subsystem, or device composed of subsidiary parts, whose joint performance determines the performance of the system in respect to one or more properties. The components of a system are those parts, at a selected level of detail, whose performance is to be related to system performance.

In particular, it is suggested that a useful approach to predictions of system performance is obtained by thinking of a complex system as organized in levels. Thus, the probability of successful performance of a complex system may be considered first as determined by the behavior of a relatively small group of major subsystems. The latter may then be analyzed in turn. This procedure should lead to efficient concentration of effort toward obtaining detailed information about subsystems which have the most important effect on system performance. With this approach, moreover, information is organized in a way which permits selection of the level of detail appropriate to the kind of question which is to be asked.

When the "system" under consideration is in fact a subsystem of a larger system, it will not always be sufficient to summarize performance by the probability of successful operation. For subsystems, the "measurement" of reliability requires a somewhat different approach—as will be seen in the discussion of the hypothetical examples given below.

Discussions of Interdependent Components

It has been observed frequently that assessments of system reliability based on the assumption that component failures occur independently of one another are approximate and usually excessively conservative. Several authors have discussed possibilities for representing the dependence of system performance on the performance of interdependent components. Some of these discussions are briefly noted here.

Statistical analysis of circuit performance has been discussed by Benner and Meredith, and Meltzer. Performance characteristics are given approximately as functions of circuit elements, so data on behavior (means, variances, and correlations) of the circuit elements can be used to predict behavior of the circuit.

If it is not possible on theoretical grounds to state the relation of system performance to component behavior, it may be possible to use multiple linear regression analysis to estimate the relationship, at least for narrow ranges of values of component characteristics. This approach has been discussed by Brown and Bear. If the estimated relationship (a linear equation) appears to give a satisfactory representation of the dependence of system performance on component characteristics, then it may be used to predict the effect on system performance of variability in component performance.

The details of these procedures will not be discussed in this paper. They are available for obtaining functional relationships between system performance and component performance in situations where their methods are applicable.

Another approach to making use of information about interdependence among components has been suggested by Elmaghraby, who considers a representation of the effect of component failures which do not directly cause system failure but increase the probability of failure of other components.

The first two main sections of this paper are devoted to discussions of the use of information relating system performance to the performance of interdependent components.

Dependence of System Performance on Component Performance

The first kind of information which is needed for relating system performance to component performance is illustrated by statements such as those discussed below. These are conditions on component behavior which permit satisfactory operation of the system. They are determined by the nature of the components and the conditions of their application in the system. These conditions are distinguished from sources of failure, which are considered later.

The system will operate only if the performance of component $A$ is satisfactory. Some components (or subsystems) have a direct and "independent" effect on system performance, in that the system cannot operate properly if these components fail to give satisfactory performance, no matter what be the performance of other components. The dependence of system performance on these components is then given by a definition as precise as possible of satisfactory performance for each of them.

References:


It should be emphasized that a component whose effect on system performance is "independent" (in the sense that the definition of satisfactory performance for the component does not involve conditions on the performance of other components) may, nevertheless, not be independent of other components when causes of failure are considered. For example, the front and rear hand-brakes of an English bicycle operate independently, and the bicycle may be said to operate satisfactorily only if both are working; but a slippery street affects the performance of both brakes. In the long run, the proportion of successful operations of the braking system will depend on the frequency of rainy days as well as on the frequency of hand-brake breakdowns.

Some components of this type have a simple go-no-go relation to system performance: if a crucial solder joint is loose, the system just cannot work. Many components, however, have a more complicated relation to system performance: some set of numerically measured characteristics of the component must remain within (known) stated bounds if the system is to operate satisfactorily.

For components having independent direct effect on system performance, the dependence of system performance on component performance may be represented on a zero-one basis. Each component either meets the stated condition or not.

The value of the system output variable \( x_o \) is determined by a known function of characteristics of a set of \( n \) components \( C_1, C_2, \ldots, C_n \). Suppose \( y_1, \ldots, y_n \) are respectively the relevant characteristics of \( C_1, \ldots, C_n \), and that it is known that \( x_o = f(y_1, \ldots, y_n) \) where the form of the function \( f(y_1, \ldots, y_n) \) is known. If the value of \( x_o \) must be in a stated range, then the values of \( y_1, \ldots, y_n \) must be restricted so that \( f(y_1, \ldots, y_n) \) is in this range. In this situation, the components \( C_1, \ldots, C_n \) are evidently interdependent; a given value of \( y_1 \) may be satisfactory when associated with one set of values for \( y_2, \ldots, y_n \) but unsatisfactory when associated with some other set. The dependence of system performance on components \( C_1, \ldots, C_n \) is given by \( f(y_1, \ldots, y_n) \), and cannot in general be stated in terms of individual conditions on the characteristics \( y_1, \ldots, y_n \).

The system will operate only if properties \( y_1 \) and \( y_2 \) of components \( C_1 \) and \( C_2 \) satisfy a specified relation. An example of such a relation is the condition that \( y_1 + y_2 \) must be greater than a specified number. The situation in this case is essentially the same as that of the preceding case.

The system output variable \( x_o \) depends on characteristics \( y_1, \ldots, y_n \) of components \( C_1, \ldots, C_n \); but the theoretical relation does not agree adequately with relations observed in practice. This situation prevails, for example, when the best available theoretical analysis of a system is based on ideal physical properties of the elements of the system, while realizations of the elements cannot have these ideal properties. (Thus, to some degree, this situation always prevails.) If it has been discovered empirically that the theoretical analysis leads to inadequate predictions of system operation, it may have been discovered also what kinds of discrepancies occur. The representation of system performance as a function of component performance must be obtained by "educated guessing." Once a mathematical expression for the reliability of the whole system has been constructed, then it can be determined by varying the conjectured relationship whether or not the accuracy of the guess will have an important effect on the accuracy of the over-all prediction. In some situations, it may be feasible to perform an experiment to estimate the empirical relation of a set of component characteristics to system performance.

The probability distribution of the system output variable \( x_o \) depends on characteristics \( y_1, \ldots, y_n \) of components \( C_1, \ldots, C_n \). This statement distinguishes a case in which the relation of system performance to component performance is statistical. In the previous cases, the probability distribution of \( x_o \) would depend on the probability distributions of \( y_1, \ldots, y_n \); this remains true, but it is supposed further that fixed values of \( y_1, \ldots, y_n \) determine not the exact value of \( x_o \) but, for example, the average value to be expected for \( x_o \). If the form of the distribution of \( x_o \) is known—e.g., if \( x_o \) has a Gaussian distribution with its mean and variance (mean-square deviation from mean) determined by known functions of the values \( y_1, \ldots, y_n \)—then the statistical relationship of \( x_o \) to \( y_1, \ldots, y_n \) is specified. This case differs from the other kinds of relationships discussed, in that there may be no values of \( y_1, \ldots, y_n \) for which it is certain that the value of \( x_o \) will be satisfactory. It will be seen, nevertheless, that it is possible (though not easy) to make predictions about the probability that \( x_o \) will have a satisfactory value.

The system will operate only if at least one of components \( A \) and \( B \) gives satisfactory performance. This is the situation if a system contains duplicate or redundant components. When the duplicate component is present on a stand-by basis to be used only if the first one fails, there may be a third component involved which detects the failure and puts the stand-by unit into operation. The representation of this kind of relationship and its effect on reliability has been discussed by Luebbert.\(^7\)

The Composition of Conditions

Suppose now, that all essential conditions on component behavior which determine satisfactory system performance are effectively accounted for. That is, suppose a set of components or subsystems have been listed together with the relation of system performance to the characteristics of each. Abstractly, the situation at this point may be illustrated by the following example.

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Fig. 1 is a diagram of a simple system $S$ whose satisfactory performance (apart from catastrophic failure) is defined in terms of the output variable $x_0$. The system $S$ is composed of three subsystems $C_1, C_2, C_3$ with outputs $y_1, y_2, y_3$, respectively. The subsystem $C_2$ is further analyzed in terms of components $D_1$ and $D_2$. The system $S$ is said to be giving satisfactory operation as long as

$$a \leq x_0 \leq b,$$

where $a$ and $b$ are given numbers. The dependence of system performance on the performance of the subsystems $C_1, C_2, C_3$ is given (say) by the following statements:

1) If $y_1 < 2$, the system $S$ fails.
2) If $y_1 + y_2 > 10$, the system $S$ fails.
3) $x_0 = K e^{t_0}.$
4) If any one of $C_1, C_2, C_3$ has a catastrophic failure, the system $S$ fails.

Thus the probability $P$ that $S$ will give satisfactory performance is given by

$$\text{Prob. } \{ C_1 \text{ does not fail and } C_2 \text{ does not fail and } C_3 \text{ does not fail and } y_1 \geq 2 \text{ and } y_1 + y_2 \leq 10 \text{ and } \log (a/K) \leq y_2 \leq \log (b/K) \}. $$

It is clear that the conditions connected by "and" in this expression are not all independent. One kind of dependence among components is explicitly stated in the condition "$y_1 + y_2 \leq 10\)." Another kind of dependence is suggested by the restriction between the conditions "$C_1$ does not fail" (i.e., no catastrophic failure) and "$y_1 \geq 2\) (i.e., no wearout failure—that is to say, the output of $C_1$ is at a satisfactory level). Further consideration of this sort of interdependence among components is deferred until the next section of this paper.

Continuing with the analysis of the system of Fig. 1, consider now the subsystem $C_2$ with two components $D_1$ and $D_2$. Suppose the dependence of $C_2$ on the performance of $D_1, D_2$, and their output variables $s_1, s_2$ is given as follows.

1) If both $D_1$ and $D_2$ fail, the subsystem $C_2$ fails.
2) The value of $y_2$ is uniformly distributed in the interval $s_0 \pm 1$, where $s_0$ is the larger of $s_1, s_2$.

Restating condition 2), it is supposed that the value of $y_2$ is not precisely determined by $s_0$, but will be within $\pm 1$ of $s_0$ with any value in this range being as likely as any other. If it is desired to expand the expression for $P$ given above, then the condition "$C_2$ does not fail" is replaced by "$D_1$ does not fail or $D_2$ does not fail." Furthermore, letting $U(z_0)$ denote a random variable uniformly distributed on the interval $s_0 \pm 1$, the condition "$y_1 + y_2 \leq 10\) is replaced by "$y_1 + U(z_0) \leq 10\).

This illustrative example will be considered further below, to illustrate the incorporation of additional information about interrelations among the components of $S$. The next main section of the paper deals with interdependence among components with respect to their modes of failure.

**INTERDEPENDENCE OF COMPONENT FAILURES**

Additional information which will be useful for assessing the relation of system performance to the performance of interdependent components is illustrated by statements such as those discussed in this section. These statements are typical of some possible interrelationships among components, arising from the nature of the application of the components in the system, and from the response of component performance to the conditions under which the system is used.

**Components A, B, and C will all fail at once if certain unusually extreme atmospheric conditions prevail; under normal conditions, they suffer other kinds of catastrophic failures or wearout failures independently of one another.**

This situation may be described by the term "conditional independence." Suppose, for instance, that a system is used in such a way that it is subjected to some kind of severe shock from time to time, and has been designed to withstand such shocks unless they are unusually severe. Prediction of system performance may be more realistic if this particular kind of failure is treated separately. The calculations which are appropriate in the conditional independence situation are illustrated by a simple numerical example.

Consider two components $A$ and $B$. Suppose that each one can exhibit three kinds of performance: satisfactory, unsatisfactory because of wearout, or failure, due to unusual shock. Suppose further that it has been established that the probabilities of each of these have been determined to be as follows, where $p_1 + q_1 + r = 1$, $p_3 + q_3 + r = 1$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Satis.</th>
<th>Unsatis.</th>
<th>Shock</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$p_a = 0.980$</td>
<td>$q_a = 0.010$</td>
<td>$r = 0.010$</td>
</tr>
<tr>
<td>$B$</td>
<td>$p_b = 0.985$</td>
<td>$q_b = 0.005$</td>
<td>$r = 0.010$</td>
</tr>
</tbody>
</table>

The probability $r$ of shock failure is the same for each component, since it depends not on the nature of the...
particular component but on the occurrence of an unusual ambient condition. One way to look at the joint performance of components \( A \) and \( B \) is to consider the joint probability distribution, which gives for every pair (performance of \( A \), performance of \( B \)) the probability of its occurrence. The following table gives the joint distribution for components \( A \) and \( B \), assuming conditional independence.

**Example 1: Joint Probabilities of Performance**

<table>
<thead>
<tr>
<th>Performance of ( A )</th>
<th>Performance of ( B )</th>
<th>Shock (r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satis. (( p_a ))</td>
<td>Satis. (( p_b ))</td>
<td>0</td>
</tr>
<tr>
<td>( 1 - r )</td>
<td>( 1 - r )</td>
<td></td>
</tr>
<tr>
<td>( 0.9751 )</td>
<td>( 0.0049 )</td>
<td>0</td>
</tr>
<tr>
<td>Unsatis. (( q_a ))</td>
<td>Unsatis. (( q_b ))</td>
<td>0</td>
</tr>
<tr>
<td>( 1 - r )</td>
<td>( 1 - r )</td>
<td></td>
</tr>
<tr>
<td>( 0.0099 )</td>
<td>( 0.0001 )</td>
<td>0</td>
</tr>
<tr>
<td>Shock (r)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( r = 0.0100 )</td>
<td></td>
</tr>
</tbody>
</table>

It is seen that the quantities in each row add up to the probability of the corresponding performance of component \( A \), and that the columns add up to the corresponding probabilities for component \( B \). The sum of all the entries is unity. The probabilities may be derived from the conditional independence assumption by the rules for calculating with conditional probabilities.\(^8\) Let \( P(U \mid V) \) denote “the probability of \( U \) on condition that \( V \) is true.” Then, for example,

\[
P(A \text{ satis. and } B \text{ satis.}) = P(A \text{ satis. and } B \text{ satis.} \mid \text{ shock occurs}) \times P(\text{shock occurs}) + P(A \text{ satis. and } B \text{ satis.} \mid \text{ no shock}) \times P(\text{no shock}).
\]

Now the first conditional probability is obviously zero; if an unusual shock occurs, both \( A \) and \( B \) fail. Thus, only the second term must be evaluated. Since \( A \) and \( B \) are assumed to be independent under normal conditions (no shock), the second conditional probability may be written as the product of two conditional probabilities,

\[
P(A \text{ satis. and } B \text{ satis.} \mid \text{ no shock}) = P(A \text{ satis.} \mid \text{ no shock}) \times P(B \text{ satis.} \mid \text{ no shock}).
\]

Evaluating these conditions from the information given about the probabilities for the individual components, it follows that

\[
P(A \text{ satis.} \mid \text{ no shock}) = p_a/(1-r),
\]

\[
P(B \text{ satis.} \mid \text{ no shock}) = p_b/(1-r).
\]

Finally, the probability of no shock is \( (1-r) \). Thus,

\[
P(A \text{ satis. and } B \text{ satis.}) = \frac{p_a}{1-r} \cdot \frac{p_b}{1-r} \cdot (1-r) = \frac{p_ap_b}{1-r}.
\]

The prediction of system reliability obtained from a joint distribution of this kind will be somewhat different from a prediction made on the assumption that the performance of the two components is unconditionally independent. To illustrate this, consider two cases. Case I: Both \( A \) and \( B \) must give satisfactory performance in order that the system operate correctly. Case II: At least one of \( A \) and \( B \) must give satisfactory performance. If it is assumed that the two components are unconditionally independent, then the probability that the joint performance of \( A \) and \( B \) will be satisfactory is calculated from the given probabilities \( p_a \) and \( p_b \) by well-known methods.\(^7\)

- **Case I:** \( P(A \text{ satis. and } B \text{ satis.}) = p_ap_b \).
- **Case II:** \( P(A \text{ satis. or } B \text{ satis.}) = p_a + p_b - p_ap_b \).

Under the conditional independence situation, the Case I probability may be read directly from the joint distribution table. The Case II probability may be calculated from the table by the usual formula:

\[
P(A \text{ satis. or } B \text{ satis.}) = P(A \text{ satis.}) + P(B \text{ satis.}) - P(A \text{ satis. and } B \text{ satis.}) = p_a + p_b - p_ap_b/(1-r).
\]

The numerical comparison is as follows:

**Probability That Joint Performance of \( A, B \) Is Satisfactory**

<table>
<thead>
<tr>
<th></th>
<th>Independence</th>
<th>Conditional Independence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case I (series)</strong></td>
<td>0.9653</td>
<td>0.9751</td>
</tr>
<tr>
<td><strong>Case II (parallel)</strong></td>
<td>0.9997</td>
<td>0.9899</td>
</tr>
</tbody>
</table>

It is true in general that if the present kind of dependence situation prevails, reliability predictions based on the independence assumption will be overconservative for components connected in series, but overoptimistic for components connected in parallel. For Case I, the differences between the two predictions will be greater if there are more than two components. For Case II (parallel), increasing the number of components makes the probability of satisfactory joint performance become close to unity very rapidly under unconditional independence; while under conditional independence the upper bound for the probability of satisfactory joint performance is \( (1-r) \).

If a system contains \( n \) identical components, connected in series, and the probability of satisfactory performance for each component is \( p \) while the probability of a “shock failure” is \( r \), then under the conditional independence assumption the probability that the whole set of \( n \) components will give satisfactory performance is

\[
p^n/(1-r)^{n-1}.
\]

Thus, suppose an equipment such as a computer contains 1000 components each of whose failure probabilities \( (1-p) \) is 1/10,000. And suppose the “shock failures”...
which affect all components simultaneously are about 10 per cent of these, i.e., $r = 10^{-4}$. The predicted frequency of failures calculated by the product rule would be

$$1 - p^n = 0.095$$

while the prediction based on the conditional independence situation would be

$$1 - p^n/(1 - r)^{n-1} = 0.086.$$  

As the performance of component $A$ deteriorates, the load on component $B$ is decreased and degradation of performance of $B$ is slowed down. For example, suppose the following extreme situation prevails. If component $A$ wears out, then component $B$ will surely continue to give satisfactory performance; and vice versa. (For simplicity, it is assumed that catastrophic failures are impossible.) This situation is a case of “negative dependence.” Let the probabilities for $A$ and $B$ be given as follows, where $p_A + q_A = 1$, $p_B + q_B = 1$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Satis. $p_A$</th>
<th>Unsatis. $q_A$</th>
<th>Satis. $p_B$</th>
<th>Unsatis. $q_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.98</td>
<td>0.02</td>
<td>0.97</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Under the assumption of negative dependence described above, the joint probability distribution for the performance of components $A$ and $B$ is given by the following.

**Example 2: Joint Probabilities of Performance**

<table>
<thead>
<tr>
<th>Performance of $A$</th>
<th>Performance of $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satis. $p_A$</td>
<td>Unsatis. $q_A$</td>
</tr>
<tr>
<td>$p_a + p_b = 0.95$</td>
<td>$q_a + q_b = 0.05$</td>
</tr>
<tr>
<td>$q_a = 0.02$</td>
<td>$q_b = 0.03$</td>
</tr>
</tbody>
</table>

These probabilities are derived from the assumptions that

$$P(A \text{ satis.} \mid B \text{ unsatis.}) = 1$$

$$P(B \text{ satis.} \mid A \text{ unsatis.}) = 1.$$  

Thus, for instance,

$$P(A \text{ satis. and } B \text{ unsatis.})$$

$$= P(A \text{ satis.} \mid B \text{ unsatis.}) \times P(B \text{ unsatis.})$$

$$= p_A = p_B.$$

To see the effect of this kind of dependence on reliability prediction, consider again the two types of possible application of components $A$ and $B$ in a system: Case I (series) and Case II (parallel). The following shows the difference between predictions made on the basis of the independence assumption and the predictions appropriate in the present type of negative dependence situation.

<table>
<thead>
<tr>
<th>Probability that Joint Behavior of $A$, $B$ Is Satisfactory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Case I (series) $A$ and $B$ satis.</td>
</tr>
<tr>
<td>Case II (parallel) $A$ or $B$ satis.</td>
</tr>
</tbody>
</table>

Observe that in this situation, the independence assumption tends to give an overoptimistic prediction in Case I, and to underestimate the effectiveness of redundant components (Case II). This is the opposite of the previous (conditional independence) situation, where the dependence between the two components was positive (shock failures occurred simultaneously in both components). Failure reports from the field have indicated that every time component $A$ fails, component $B$ fails also. This statement illustrates a particular kind of positive dependence which may be called “chain dependence.” Failure of component $A$ always leads to failure of component $B$, while component $B$ may be subject to additional types of failure.

**Example 3: Joint Probabilities of Performance**

<table>
<thead>
<tr>
<th>Performance of $A$</th>
<th>Performance of $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satis. $p_A$</td>
<td>Unsatis. $q_A$</td>
</tr>
<tr>
<td>$p_a = 0.01$</td>
<td>$q_a = 0.02$</td>
</tr>
</tbody>
</table>

As usual, $p_a + q_a = 1$, $p_b + q_b = 1$. Once the zero has been inserted in the cell corresponding to the type of joint performance which is assumed to be impossible, the remaining entries shown above are determined by the requirement that the rows and columns each add up to the appropriate individual probabilities. The following computation provides a comparison between the assumption of independence and the assumption of chain dependence as given in the table. Let $q_a = 0.01$, $q_b = 0.02$.

**Probability of Successful Joint Performance**

$$(A \text{ and } B \text{ satis.})$$

<table>
<thead>
<tr>
<th></th>
<th>Independence</th>
<th>Chain Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_a p_b = 0.9702$</td>
<td>$p_a = 0.9800$</td>
</tr>
</tbody>
</table>

The three examples given in this section of the paper were intended to illustrate some of the possibilities for obtaining mathematical expressions which incorporate information about the occurrence of failures. In the next section, some additional assumptions are made about the system $S$ of Fig. 1, and the calculations for that example are continued.

**Some Simple Hypothetical Examples**

**First Illustrative System, System $S$**

Consider again the system $S$ of Fig. 1, and suppose that it is desired to make use of some additional informa-
tion about interdependence among failures of the components of \( S \).

**Shock Failure:** Analysis of component properties and of the proposed conditions of use for the system \( S \) have indicated that all components will fail simultaneously if the system is exposed to a certain extreme atmospheric condition. The probability of this occurrence is \( r \), and joint occurrence of this shock situation and other failures is impossible. Then (cf., Example 1 of the preceding section) all remaining calculations are understood to be valid on condition that no shock has occurred. The general rule is

\[
Pr (U \text{ and } V) = Pr (U \mid V) \times Pr (V).
\]

Accordingly, the probability \( P \) of successful system performance is obtained from the probability of successful system performance on condition that no shock has occurred, by multiplying the latter by (1 - \( r \))—the probability that no shock occurs.

**Component \( C_2 \):** Recall that this component is in fact a subsystem containing two components \( D_1 \) and \( D_2 \). Suppose that (on condition no shock has occurred) the components \( D_1 \) and \( D_2 \) are independent and have the same performance probabilities. Let \( q_0 \) be the probability of a catastrophic failure (other than shock failure) for \( D_1 \) or \( D_2 \) and let the probability distributions for the output variables \( z_1, z_2 \) be \( f_{D_1}(z_1), f_{D_2}(z_2) \), respectively (when no catastrophic failure has occurred).

Suppose that, except for shock failure, failures of component \( C_2 \) occur independently of the performance of components \( C_1 \) and \( C_3 \).

**Components \( C_1 \) and \( C_3 \):** Suppose component \( C_3 \) is believed to be chain-dependent on component \( C_1 \) (cf., Example 3 of the preceding section). In particular, let \( q_{01} \) and \( q_{02} \) be the respective probabilities of catastrophic failure (on condition of no shock), and suppose the joint probabilities of catastrophic failure are as follows.

<table>
<thead>
<tr>
<th>Component ( C_1 )</th>
<th>Component ( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Failure ((1-q_{01}))</td>
<td>Failure ( q_{01} )</td>
</tr>
<tr>
<td>No failure ((1-q_{02}))</td>
<td>Failure ( q_{02} )</td>
</tr>
</tbody>
</table>

Suppose further that if no catastrophic failure has occurred \( C_1 \) and \( C_3 \) are independent. That is, the values of the output variables \( y_1 \) of \( C_1 \) and \( y_2 \) of \( C_3 \) have independent probability distributions \( f_{C_1}(y_1), f_{C_2}(y_2) \) when no catastrophic failure has occurred to either one.

**Expression for \( P \):** Recall now the expression for the probability \( P \) that the system \( S \) will give satisfactory performance. \( P \) was expressed as the probability of joint occurrence of several events. With the assumptions now given, \( P \) can be expanded by repeated applications of the rule

\[
Pr (U \text{ and } V \mid W) = Pr (U \mid V \text{ and } W) \times Pr (V \mid W).
\]

The result is

\[
P = (1 - r) \times (1 - q_0) \times (1 - q_{01}) \times Pr (y_1 \geq 2 \text{ and } y_1 + U(z_2) \leq 10 \mid Q) \times Pr \left( \log \frac{a}{K} \leq y_2 \leq \log \frac{b}{K} \mid Q \right) + 2(1 - r) \times q_0 \times (1 - q_{02}) \times Pr (y_1 \geq 2 \text{ and } y_1 + U(z_1) \leq 10 \mid Q^*) \times Pr \left( \log \frac{a}{K} \leq y_2 \leq \log \frac{b}{K} \mid Q^* \right)
\]

where \( Q \) stands for the condition “\( C_1 \) does not fail and neither \( D_1 \) nor \( D_2 \) fails and \( C_3 \) does not fail;” \( Q^* \) denotes the condition “\( C_1 \) does not fail and exactly one of \( D_1 \), \( D_2 \) fails and \( C_3 \) does not fail.”

Now, \( Pr \left( \log \frac{a}{K} \leq y_2 \leq \log \frac{b}{K} \mid Q \right) \) may be calculated from the probability distribution \( f_{C_1}(y_2) \)—and is the same whether the condition is \( Q \) or \( Q^* \). Under condition \( Q \), the probability of “\( y_1 \geq 2 \) and \( y_1 + U(z_2) \leq 10 \)” can in principle be calculated from the probability distributions \( f_{C_1}(y_1), f_{D_2}(z_1), \) and \( f_{D_2}(z_2) \). The distribution of \( z_2 = \max (z_1, z_2) \) is determined first; from this, the distribution of \( U(z_2) \) is obtained. Under condition \( Q^* \), exactly one of components \( D_1, D_2 \) has not failed, so that \( z_2 \) is equal to \( z_1 \) (say). The distribution of \( z_2 \) is then simply \( f_{D_1}(z_1) \), and the distribution of \( U(z_2) \) is obtained from it. The second term of the expression for \( P \) is multiplied by 2 because the condition \( Q^* \) can occur in two ways (\( D_1 \) fails or \( D_2 \) fails) while the probabilities are the same for both.

It may be seen that the expression for \( P \) has now been expanded into a form which can be calculated when the following data are available: 1) estimates of the probability distributions for output variables \( y_1, y_2, z_1, z_2 \) on condition that no catastrophic failure occurs; 2) an estimate of the probability of shock, \( r \); 3) an estimate of \( q_{01} \), the probability of catastrophic failure of \( D_1 \) (or \( D_2 \)) on condition that no shock has occurred, and 4) an estimate of \( q_{02} \), the probability of catastrophic failure of \( C_2 \) on condition no shock has occurred.

**Second Illustrative System, System \( R \):**

The preceding example was constructed to illustrate the possibility of using various types of information together. The present example has less variety, with a larger number of components. Suppose a system \( R \) is composed of two identical major subsystems \( R_1 \) and \( R_2 \) with output variables \( x_1 \) and \( x_2 \). The output \( x \) of \( R \) is satisfactory if \( x \geq a \), where \( a \) is a fixed number; and \( x \) equals the larger of \( x_1 \), \( x_2 \). The subsystem \( R_1 \) is composed of a power source which is either working or not; and 100 identical components, each one of which is either working correctly, or not. The output \( x_1 \) of \( R_1 \) is a function \( x_1 = f(n_1) \) of the number \( n_1 \) of components of \( R_1 \) which are working. Similarly, \( x_2 = f(n_2) \).

There is a possibility of shock failure (probability \( r \)) which would affect all components at once. In the ab-
ence of shock failure, $R_3$ and $R_5$ are negatively dependent with respect to the occurrence of power failures (say there is always exactly one auxiliary power source available). The probability that a power source fails is $q$. In the absence of shock or power failures, all components of $R_1$ and $R_2$ are independent and each has (conditional) probability $p$ of working.

With this information, the distribution of $x_1 = f(n_1)$ may be calculated from the binomial probability distribution which governs the performance of the 100 components of $R_1$ in the absence of shock or power failure. This calculation requires only an estimate of $p$ [and of course knowledge of the form of the function $f(n_1)$].

Given the (identical) distributions of $x_1$ and $x_2$, the distribution of $x = \max (x_1, x_2)$ on condition of no shock or power failure can be calculated. An expression for $P$ is then obtained as follows.

$$P = (1 - r)(1 - 2q) \Pr \{ \max (x_1, x_2) \geq a | \text{no shock,}
\text{either } R_1 \text{ nor } R_2 \text{ has power failure} \}
+ 2(1 - r)q \Pr \{ x_1 \geq a | \text{no shock, } R_2 \text{ has power failure but } R_1 \text{ does not} \}.$$

This expression can be evaluated if estimates are available for $p$, $q$, and $r$.

Discussion of Examples

In both of the examples just given, it is seen that the performance of a subsystem or component is not always summarized by the "probability of successful component performance." In this respect, the analysis of a whole system may frequently differ from the analysis of a subsystem. For instance, for the second hypothetical system, it is desirable to know in detail the conditional probability distribution of the output variable $x_1$ (on condition of no shock and no power failure). Indeed, "$\Pr \{ \text{subsystem } R_0 \text{ operates satisfactorily} \}^*$ does not have a useful definition, since this probability depends on the performance of $R_2$.

It is suggested that analysis of subsystems is most usefully summarized by estimates of the probability distributions of its output variables under various conditions.

Confidence Interval for $P$

The preceding discussion has been directed toward the construction of a mathematical model of a system, incorporating information about interdependence among components. The outcome was an expression for the probability $P$ of satisfactory system performance which could (in principle at least) be evaluated from estimates or conjectures for certain probabilities and conditional probability distributions. If the assumptions relating system performance to component performance were essentially correct, then the value of $P$ could be predicted.

Suppose, however, that, as usual, only a limited number of components are available for making tests to estimate the required performance probabilities. Then, because of variability among components, the estimates are not very precise. It is desirable in such a situation to obtain also an interval estimate for $P$, so that the width of the interval can suggest how much uncertainty attaches to the estimated value of $P$. Such an interval estimate is called a confidence interval. 1

A confidence interval for $P$ is determined by a confidence limit $P_0$, smaller than the estimated value of $P$, which is derived by a method which insures that the statement "$P \geq P_0$" can be made at a specified confidence level (often 95 or 99 per cent). The confidence limit $P_0$ is determined by (1) the preassigned desired confidence level and 2) assumptions which have been made about the underlying probability distributions of the measurements which have been made to obtain an estimate of $P$. A fairly strict interpretation of a confidence interval is that if the data collection procedure could be repeated over and over, and if the same procedure were used each time to calculate (say) a 95 per cent confidence limit $P_0$; then in the long run the statement "$P \geq P_0$" made after each repetition of the procedure would be true 95 per cent of the time.

A looser interpretation of the statement, "$P \geq P_0$ at the 95 per cent confidence level," is that $P_0$ is the smallest value which is not extraordinarily unlikely in view of the measurements which have been made.

Illustration, for the System $R$

The usefulness of an interval estimate can be illustrated quite sharply in the context of the second illustrative example (the system $R$) of the preceding section.

In order to estimate $P$ for the system $R$, it is necessary to estimate three probabilities:

- $r =$ Probability of shock failure.
- $q =$ Probability of power failure in subsystem $R_1$ (or $R_0$) on condition no shock occurs.
- $p =$ Probability that one of the 100 components of $R_1$ (or $R_0$) works correctly on condition of no shock and no power failure.

Suppose, for simplicity, that only negligible uncertainty attaches to the estimates of $r$ (based on a long history of weather data) and of $q$ (based on long experience with the particular type of power source involved). Attention is focused on the problem of estimating $p$.

Tests have been made on 1000 components of the type to be used in the subsystems $R_1$ and $R_0$. None of them has failed to work correctly. Nevertheless, although this component is believed to be highly reliable, it is not believed to be perfect. Assuming that the "true" long-run proportion of satisfactory components of this type is $p$ (less than unity), how small might $p$ be without its being extraordinarily unlikely that every one of a sample of 1000 did not fail? The following table gives several answers to this question, corresponding to various confidence levels (i.e., various meanings of "extraordinarily unlikely"). It also gives confidence limits for the proba-
bility $p$ for sample sizes 500 and 5000, when no failures have occurred among the sample items.

<table>
<thead>
<tr>
<th>Confidence Level (In Per Cent)</th>
<th>Sample Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>0.9943</td>
</tr>
<tr>
<td>99</td>
<td>0.9908</td>
</tr>
<tr>
<td>99.5</td>
<td>0.9895</td>
</tr>
</tbody>
</table>

If the 99 per cent confidence level is adopted, then—since the sample size was 1000—the estimated confidence limit for $p$ would be $p_{99}=0.9954$. The ordinary (point estimate) for the probability $P$ for system $R$ would be obtained using $p=1$; the only predicted failures would be shock and power failures. An interval estimate for $P$ would be obtained by using $p_{99}$ instead of $p=1$; the result of this calculation would be a confidence limit $P_{99}$ for $P$.

The foregoing is admittedly a relatively simple example of a situation where a confidence limit for $P$ could be calculated. In many cases, various approximations would be required, as well as difficult numerical integrations. The point is, that the calculation of a confidence limit for $P$ is in principle possible once there exists a mathematical expression relating $P$ to measurable aspects of component performance. And the calculation of a confidence interval provides reasonable protection against the imprecision of estimates based on relatively small samples.

**Illustration, for the System S**

Suppose that, for the system $S$ of Fig. 1, the following confidence intervals have been obtained, each at the 99.5 per cent confidence level.

$(1-r_3) \geq (1-r_0)$—probability of no shock,

$(1-q_{g_3}) \geq (1-q_{g_0})$—probability $C_3$ does not fail (on condition no shock),

$(1-g_{p_3}) \geq (1-g_{p_0})$—probability $D_3$ does not fail (on condition no shock).

Suppose further that extensive experiments have verified that it is reasonable to assume that the (conditional) probability distributions of $y_1, y_2, z_1, z_2$ in an absence of catastrophic failure have means at “design center,” known variances, and have the Gaussian form (but are truncated at upper and lower specification limits).

Now a method which suggests itself immediately for obtaining an approximate confidence limit $P_{99}$ for the system probability $P$, is to insert the confidence limits $(1-r_3)$, etc., in the expression for $P$. But each of the three confidence limits was obtained by a separate set of tests; each limit has confidence level 99.5 per cent, but a function of the three limits is subject to cumulative errors and must in general be less precise. Indeed, the confidence level for simultaneous assertion of the three confidence intervals is $(0.995)^3=0.990$, i.e., 99 per cent. This is a lower bound for the confidence level for asserting “$P \geq P_0$.”

If the functional form of $P$ as a function of $(1-r)$, $(1-q_{g_0})$, and $(1-g_{p_0})$ were taken into account, together with assumptions concerning the underlying probability distributions governing the measurements from which they are estimated, it would in principle be possible to improve on such a lower bound for the confidence level for asserting “$P \geq P_0$.” One way of making such an improvement in a special case has been considered by Buehler.10

**FURTHER USES OF A MATHEMATICAL EXPRESSION FOR $P$**

In this section some additional uses are noted for a mathematical expression for the system probability $P$, constructed through the suggested approach.

**Decisions About Design Changes**

Given a mathematical expression for the probability $P$, incorporating information on interdependence of the components of a system, one may raise the following types of questions.

1) What would be the effect on $P$ if a duplicate of a certain subsystem were added so that the system could operate if at least one of the subsystems did not fail?

2) Supposing that it has been decided to add redundant components, for which types of component should they be added?

3) What would be the effect of replacing a given component type by an improved type less frequently subject to catastrophic failure?

Many such questions can be answered usefully only in the context of costs: “Which is more costly, the present failure rate or the improvements required to achieve a lower failure rate?” The effect of certain kinds of changes of design may be calculated by a modification of the mathematical expression for $P$; the costs must be estimated from additional information. If several changes are proposed, each of which has presumably the same cost, they can be compared as to their effect on $P$. If it is specified that $P$ must be improved by a fixed amount at least, manipulation of a mathematical expression for $P$ can be helpful in deciding which design changes would be sufficient to accomplish this.

The approach to analysis of system performance suggested in this paper calls attention to the possibilities for isolating and determining the effect of particular types

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These values were computed directly since (in the case that no failures are observed in the sample) $p_0$ is obtained by solving for $p$ in the equation $p^n = (1-n), (1-q_{g_0}), (1-g_{p_0})$ were taken into account, together with assumptions concerning the underlying probability distributions governing the measurements from which they are estimated, it would in principle be possible to improve on such a lower bound for the confidence level for asserting “$P \geq P_0$.” One way of making such an improvement in a special case has been considered by Buehler.10

of failure which may be removable, if their effect is great enough to justify the cost of making improvements.

Simulation of the Effect of Component Specifications

The attempt is generally made, in setting component specifications, to insure that every component which meets specifications will (in the absence of catastrophic failure) give satisfactory performance. But when the performance of one component is dependent on that of another in such a way that the first may or may not be satisfactory depending on the performance of the other, it may be difficult to determine realistic specifications.

Suppose, then, that a mathematical expression for the system probability \( P \) is available, relating system performance to the values of component characteristics, and taking account of interdependence among components. Then the same calculations which are carried out to obtain predictions of \( P \) on the basis of measurements made on components could be carried out using hypothetical "measurements" obtained by assuming that the probability distribution of a characteristic of a component has some assumed form (truncated at the specification limits).

Such calculations would make it possible to make predictions as to the effect of specification changes: in one case, the probability distributions of component characteristics might be unchanged, with only the truncation points (specification limits) altered; in another case, the whole probability distribution for a component characteristic might shift so as to have its mid-point at a new "design center"; in a third case, the truncation points might shift toward the center and the variance of a component characteristic decrease also, so that about the same proportion of components would still meet specifications.

The extreme case of these calculations—all component characteristics having their values at specification limits—is often unrealistic and can lead to either excessively stringent component specifications or excessively pessimistic predictions of system reliability. It is suggested that various "simulations" of the effect of component specifications can help to determine whether a given set of specifications is realistic.

CONCLUSION

The purpose of this paper has not been to outline detailed techniques for every step of the process of predicting reliability, but rather to suggest an approach to system analysis which would organize engineering design information and data on component performance in a way suitable for the application of probability theory and of the techniques of mathematical statistics.

If assumptions relating system performance to component performance and system design and use are essentially valid, one may obtain an estimated value for the system probability \( P \) and usually at least an approximate confidence interval reflecting the imprecision of this estimate.

In the long run, of course, the accuracy of a prediction based on an estimate of \( P \) can be determined only by experience with a system as a whole. It is believed, nevertheless, that the approach adopted in this paper may help to make possible some representations of system performance as a function of interdependent component performance which are sufficiently accurate to provide useful estimates of the system reliability.

Evaluation of Failure Data

HERBERT I. ZAGOR†

GENERAL

Among the many benefits for management which can be derived from a failure evaluation program are: 1) evaluating equipment and its maintenance procedures; 2) directing and guiding reliability improvement as well as future design; and 3) estimating spares.

Failure data obtained from one equipment may be used for reliability and spares predication for similar equipments under similar environments since operational comparisons may be made directly on such items.

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Increasing amounts of data are becoming available to allow correlation factors to be determined so that one can extend these predications to diverse equipments.

The problem of stocking spare parts resolves itself into setting up on a probability basis an equilibrium condition between the expected number of failures and the number of stocked items. Once spare parts estimates are available, a logistic system can be determined.

The main assumption is that the reliability of an equipment in its steady state behavior is such that those failures which inevitably occur are independent, random, and occur at a uniform rate. The failure dis-