THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS
BY DIFFERENCE METHODS USING THE ELECTRONIC DIFFERENTIAL ANALYZER

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Abstract

Partial differential equations can be approximated by systems of simultaneous ordinary differential equations by replacing one or more of the partial derivatives by the appropriate finite differences. The resulting systems of equations can sometimes be solved directly by an electric analog computer employing passive circuits (e.g. the Caltech analog computer), or by an electronic differential analyzer employing feedback amplifiers (e.g. the Reeves Electronic Analog Computer). The latter type of computer has several important advantages, including versatility, ability to handle nonlinear partial differential equations, and flexibility in selecting time scales. Both theoretical analysis of the accuracies attainable with the difference method and actual solution examples using the electronic differential analyzer are described. Types of partial differential equations considered include the heat, wave, and beam equations. Real-time simulation of aircraft structures is discussed.

The actual computer solutions were carried out on the electronic differential analyzer of the Department of Aeronautical Engineering. As a result of the promise shown by the difference method discussed in this report, construction of an 80-amplifier analyzer has begun.

INTRODUCTION

Usual Differential Analyzer Technique for Solving Partial Differential Equations

The electronic differential analyzer is limited to the solution of ordinary differential equations. To solve a linear partial differential equation on the analyzer, it is necessary to separate variables and hence convert the partial differential equation to ordinary differential equations of the eigenvalue type. The normal modes from which the solution to the original problem can be built up must then be found, usually by trial and error techniques.
The above method of separating variables and obtaining a series type of
solution can be carried out fairly efficiently on an electronic differential
analyzer.\(^1\),\(^2\),\(^3\) Certainly, for most problems the analyzer is much faster than
any hand methods. But for the engineer who is interested in getting numerical
answers to specific problems, even the analyzer approach might seem somewhat
tedious. It therefore would be highly advantageous to be able to solve the
partial differential equations directly. This can be done by replacing some
of the partial derivatives by finite differences in order to convert the original
partial differential equation into a system of ordinary differential equations.\(^4\)

Replacement of Partial Derivatives by Finite Differences

Assume we are interested in solving a partial differential equation in which
the dependent variable \( y(x,t) \) is a function of both a distance variable \( x \) and a
time variable \( t \). Instead of measuring the variable \( y \) at all distances \( x \), let us
measure \( y \) only at certain stations along \( x \); thus, let \( y_1 \) be the value of \( y \) at the
first \( x \) station, \( y_2 \) be the value of \( y \) at the second \( x \) station, \( y_n \) be the value
of \( y \) at the \( n \)th \( x \) station. Further, let the distance between stations be a
constant \( \Delta x \).

Now clearly a good approximation to \( \frac{\partial y}{\partial x} \) (i.e., the partial derivative of
\( y \) with respect to \( x \) at the \( 1/2 \) station) is given by the difference

\[
\frac{\partial y}{\partial x}\bigg|_{x=1/2} = \frac{y_1 - y_0}{\Delta x}.
\]

(1)

In fact the limit of Equation 1 as \( \Delta x \to 0 \) is just the definition of the
partial derivative at that point. Writing Equation 1 in more general terms

\[
\frac{\partial y}{\partial x}\bigg|_{x=n-1/2} = \frac{y_n - y_{n-1}}{\Delta x}
\]

(2)

In the same way

\[
\frac{\partial^2 y}{\partial x^2}\bigg|_{n} = \frac{1}{\Delta x} \left( \frac{\partial y}{\partial x}\bigg|_{n+1/2} - \frac{\partial y}{\partial x}\bigg|_{n-1/2} \right) = \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta x)^2}
\]

(3)

Similarly

\[
\frac{\partial^3 y}{\partial x^3}\bigg|_{n-1/2} = \frac{y_{n+1} - 3y_n + 3y_{n-1} - y_{n-2}}{(\Delta x)^3}
\]

(4)

Thus we have converted partial derivatives with respect to \( x \) into algebraic
differences. The only differential needed now is with respect to the time
variable \( t \), so that we are left with a system of ordinary differential equations
involving dependant variables \( y_0(t), y_1(t), \ldots, y_n(t), \ldots \).
SOLUTION OF THE HEAT EQUATION

Equation to be Solved

As a first example of a partial differential equation, let us consider the equation of heat flow through a continuous medium, since it involves second order spatial derivatives and only first order time derivatives. The basic heat equation is given by

$$\partial u / \partial t = \nabla \cdot (k \nabla u) + f \tag{5}$$

where

- \( u \) = temperature and is a function of the spatial coordinates and time,
- \( k \) = thermal conductivity, in general a function of the spatial coordinates,
- \( C \) = specific heat, a function of spatial coordinates,
- \( \delta \) = density, also a function of spatial coordinates,
- \( t \) = time,
- \( f \) = rate of heat supplied by sources in the medium, a function of spatial coordinates and time.

The actual heat flow or flux due to conduction normal to any unit surface is given by \(- k \nabla u\) (component of \( \nabla u \) normal to the surface). Thus the heat flux \( F_x \) across a unit surface normal to the x direction is given by

$$F_x = -k \frac{\partial u}{\partial x} \tag{6}$$

In a given heat flow problem it is necessary to stipulate spatial boundary conditions either on the temperature \( u \) or the heat flow \(-k \nabla u\), as well as initial temperature distribution throughout the medium.

Derivation of the Difference Equations

For simplicity in illustrating the application of difference techniques, let us assume that spatial variations in the temperature \( u \) are confined to the x direction. Equation 5 then becomes

$$C(x) \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[ k(x) \frac{\partial u(x,t)}{\partial x} \right] + f(x,t) \tag{7}$$

where we have let

$$C(x) = c(x) \delta(x) \tag{8}$$

This could represent the temperature distribution in a medium between two infinite slabs.
Following the technique discussed in Section I, we will consider only values of \( u \) at certain equally spaced stations along the \( x \) coordinate axis. Thus \( u(x,t) \) is replaced by \( u_1(t), u_2(t), \ldots \) etc. If \( \Delta x \) is the distance between stations, we can write for the heat flux \( F_{n-1/2} \) at the \( n-1/2 \) station

\[
F_{n-1/2} = -K \left. \frac{\partial u}{\partial x} \right|_{n-1/2} = -\frac{K_{n-1/2}}{\Delta x} (u_n - u_{n-1}).
\]  

We can now write the equation of heat-flow balance at the \( n \)th station. Thus

\[
c_n \frac{du_n}{dt} = \frac{K_{n+1/2}}{(\Delta x)^2} (u_{n+1} - u_n) - \frac{K_{n-1/2}}{(\Delta x)^2} (u_n - u_{n-1}) + f_n
\]  

where \( c_n \) is the heat capacity at the \( n \)th station and \( f_n \) is the rate of heat supplied by a heat source at the \( n \)th station (\( f_n \) will in general be a function of time).

Note that \( du_n/dt \) is now a total derivative and not a partial derivative, since by definition \( x \) remains fixed while we take \( du_n/dt \).

Equation 10 will be iterated for different values of \( n \) until the boundaries in \( x \) are reached, at which point it is necessary to impose boundary conditions.

**Imposing Boundary Conditions**

Suppose that one of the boundary conditions specifies the temperature at \( x = 0 \) (i.e., at the zero station). Then we have

\[
u_0 = \text{constant}
\]

and hence

\[
F_{1/2} = -\frac{K_{1/2}}{x} \left[ u_1(t) - u_0 \right].
\]  

(11)

All we have done in imposing the boundary condition, then, is to fix \( u_0(t) \) at a constant value of \( u_0 \).

If the temperature is specified at \( x = L \) (i.e., at the \( N \)th station, where \( N = L/\Delta x \)), then for \( u_N(t) \) we substitute \( u_N = \text{constant} \), the desired temperature.

Often a condition is placed on the rate of heat flowing past a boundary, either that this flow be zero (as for an insulating boundary) or a constant. Suppose we let

\[
F_{1/2} = \text{constant}.
\]

Then the equation for the first station is

\[
c_1 \frac{du_1(t)}{dt} = \frac{K_{3/2}}{(\Delta x)^2} \left[ u_2(t) - u_1(t) \right] + \frac{F_{1/2}}{\Delta x} + f_1(t).
\]  

(12)
The equations for \( u_2, u_3, \ldots \) are the same as usual. If we desire \( F_{N+1/2} = \text{constant} \) as a boundary condition, then the equation for the \( N \)th station is similar.

The process of setting in boundary conditions is evidently quite straightforward. Notice, however, that when we denote temperature at integral stations, the boundary occurs at an integral station when temperature at the boundary is specified, whereas the boundary occurs at a half-integral station when the heat flow at the boundary is specified.

**Imposing Initial Conditions**

In addition to specifying boundary conditions in this type of heat problem, it is necessary to specify the initial temperature distribution in our medium. Thus we have

\[
\begin{align*}
    u_1(0) &= u_1 \\
    u_2(0) &= u_2 \\
    u_3(0) &= u_3 \\
    & \vdots \\
    u_N(0) &= u_N
\end{align*}
\]  

(13)

These initial conditions must then be imposed on the electronic differential analyzer.

**Complete Differential Difference Equations for a Given Set of Boundary Conditions**

For purposes of illustration, let us assume that the boundary conditions of our conducting slab are that at \( x = 0 \) the temperature remains fixed at \( u_0 \), and at \( x = L = \Delta x(N+1/2) \) the heat flow is zero. The space between \( x = 0 \) and \( x = L \) is therefore broken into \( N \) cells, and from Equations 10, 11, and 12 we can write the complete set of differential equations.

The initial conditions specify the temperature for each station at \( t = 0 \). A schematic diagram showing all the locations relative to the conducting slab is shown in Figure 1 for \( N = 10 \).

In Figure 2 the computer arrangement for solving the difference equation is shown. Note that the outputs of each successive row of amplifiers are reversed. This allows the necessary differences to be taken without sign-reversing amplifiers. Note also that the heat flow or flux \( F \) is available at any half-station as a dependent variable. Thus the temperature \( u \) and heat flux \( F \) across the slab can be observed directly as a function of time.
It is possible to reduce the number of amplifiers from three to one per station. In many ways, however, the circuit of Figure 2 is simpler despite the increased number of amplifiers. To change the conductivity K or heat capacity C at any station, only the appropriate resistor has to be varied. Initial temperature distribution across the slab is changed by setting the $U_1, U_2, \ldots U_N$ voltages to the desired values. The heat sources through the slab are represented by the voltages $f_1, f_2, \ldots f_N$ which may be varied as a function of time in any desired manner.

Solution by Separation of Variables

In order to evaluate the accuracy of the difference technique, it is worth while to solve the partial differential equations of heat flow by separating variables. For simplicity we will solve the problem of the temperature distribution between two infinite slabs held at a temperature of zero. Assume that the medium has constant conductivity $K$ and constant specific heat capacity $c$. Also assume no heat sources within the medium. Then from Equation 7

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{K} \frac{\partial^2 u}{\partial x^2} \quad (14)$$

The boundary conditions are

$$u(0,t) = u(L,t) = 0 \quad (15)$$

Let us assume as a simple initial condition that the temperature in the medium is everywhere constant at $t = 0$. Thus

$$u(x,0) = U = \text{constant} \quad (16)$$

The complete solution can be written in the series form

$$u(x,t) = \frac{4U}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{(n)\pi x}{L} e^{-\frac{(n)^2\pi^2}{L^2} t} \quad (17)$$

This solution actually represents an infinite number of sinusoidal temperature distributions across the medium from $x = 0$ to $x = L$. At $t = 0$ the sine waves all add up to give the initial flat temperature distribution. For $t > 0$ the sine waves decay exponentially at different rates, with the decay rate faster for those sine waves having more nodes and loops. The resulting temperature distribution at various times is plotted in Figure 3, where a dimensionless time variable $\gamma$ has been used. $\gamma$ is defined as

$$\gamma = \frac{K}{c\delta^2} t \quad (18)$$

Thus Figure 3 is independent of the physical constants of the problem.
We will now proceed to calculate the decay-time constants for the normal modes of the difference equation representation. If these agree well with the decay-time constants in the solution above, and if the equivalent normal modes show good agreement with sine waves, then we can expect accurate results using the difference technique.

Solution of the Difference Equation for N Cells

When the space between \( x = 0 \) and \( x = L \) is broken up into \( N \) cells, the general difference equation is given by Equation 10. At station 1 and station \( N-1 \) the difference equation is obtained by setting \( u_0 \) and \( u_N \) equal to zero respectively. In the problem under consideration the conductivity \( K \) and specific heat capacity \( C \) are constant. By proper choice of our distance variable \( x \) we can make \( \Delta x = L/N \), so that \( L = N \Delta x = N \). By proper choice of our time variable \( t \) we can make \( C/K = 1 \) so that for \( f = 0 \) Equation 10 becomes for the \( i \)th cell,

\[
\frac{u_i}{u_{i+1}} = u_{i+1} - 2u_i + u_{i-1}.
\]  

(19)

To solve for the normal modes we assume that the \( i \)th temperature \( u_i \) varies with time as \( a_i e^{-\lambda t} \), where \( a_i \) is a constant. If this is true, then Equation 19 becomes a set of \( N-1 \) simultaneous algebraic equations. The only nontrivial solution of the equations is obtained when the determinant of the coefficient vanishes. This determinant, when expanded, becomes a polynomial in \( \lambda \) of order \( N-1 \). The polynomial will have \( N-1 \) positive roots \( \lambda_n \) which are the decay constants for our \( N \) cell system. To solve this determinant for a specific \( N \) is very tedious, and to solve it in general would be next to impossible. The roots \( \lambda_n \) can be found much more easily by the following procedure.

Assume that the spatial mode shape for the difference equations is the same as for the continuous equation, i.e., sinusoidal. If this is true, then for the temperature \( u_i \) at the \( i \)th station we have

\[
u_i = a \sin \frac{n\pi i}{N} e^{-\lambda_n t}.
\]  

(20)

From simple trigonometry it follows that

\[
u_{i+1} + u_{i-1} = 2a \sin \frac{n\pi i}{N} \cos \frac{n\pi}{N}.
\]  

(21)

From Equation 19 we have for the \( i \)th station

\[
(2-\lambda_n) \sin \frac{n\pi i}{N} e^{-\lambda_n t} = 2 \sin \frac{n\pi i}{N} \cos \frac{n\pi}{N} e^{-\lambda_n t}
\]

from which

\[
\lambda_n = 2(1 - \cos \frac{n\pi}{N}).
\]  

(22)
It is easy to show that Equation 20 satisfies the boundary conditions. Thus, our assumed solution is the exact solution, where the decay constants $\lambda_n$ are given by Equation 22. Expanding Equation 22 in a power series, we have

$$\lambda_n = \left(\frac{2\pi n}{N}\right)^2 \left[1 - \frac{1}{12}\left(\frac{2\pi n}{N}\right)^2 + \cdots\right].$$

(23)

In the limit of infinitely many cells the $\lambda_n$ equation given above reduces to the decay constants in Equation 17, since here $L = N \Delta x = N$ and $\frac{K}{c^2} = 1$.

In Figure 4 the percentage deviation in decay constant due to the difference method as a function of the number of cells $N$ is shown. Note that the lower modes (lower values of $n$) require fewer cells to give accurate decay constants.

To summarize, we see that when the spatial derivatives of the heat equation are replaced by finite difference, the resulting normal mode shapes agree exactly, whereas the decay constants (eigenvalues) for each mode are somewhat smaller. This means that the higher modes will decay somewhat slower when the differential difference equation approximation is used. The error is bigger for higher modes, but fortunately the higher modes are generally much less important.

**Computer Solution for One-Dimensional Heat Flow**

We now proceed to the computer solution of the one-dimensional heat flow problem considered in the last section, namely the temperature distribution between two infinite slabs a distance $L$ apart and with boundaries held at zero temperature. We can select the distance variable so that $\Delta x = 1$ and hence $L = N \Delta x = N$, where $N$ is the number of cells. After proper choice of the units of time $t$ so that $\frac{K}{c^2} = 1$, the basic heat equation becomes from Equation 14

$$\frac{2u}{t} = \frac{\partial^2 u}{\partial x^2}$$

(24)

which in terms of a difference equation is

$$\frac{du}{dt} = u_{n+1} - 2u_n + u_{n-1}.$$ 

(25)

For the problem in Section II the initial temperature distribution was a constant $U$. Thus, we have the initial conditions

$$u_n(0) = U = \text{constant}.$$ 

(26)

Boundary conditions are

$$u_0 = u_N = 0.$$ 

(27)

This heat problem was solved with the differential analyzer for 9 cells.
In order to compare the computer results with the solution shown in Figure 3 for a continuous medium, we must convert our computer time units to the dimensionless units of Figure 3. Remembering that we chose computer time units so that \( K/\rho c = 1 \), we have from Equation 18

\[
\gamma = \frac{1}{L^2} t = \frac{1}{N^2} t
\]  

(28)

Thus for our 9-cell problem we divide computer time \( t \) by 81 to obtain the dimensionless time \( \gamma \) of Figure 3. In this way points from the computer solution are compared in Figure 3 with the theoretical solution for a continuous medium. The correlation is evidently quite good, as we could have predicted from our theoretical work in Section II.

Since our initial temperature distribution is symmetric about the station 4-1/2, as are our boundary conditions, the temperature distribution remains symmetrical as a function of time. Therefore, the heat flow will be zero at station 4-1/2, and the appropriate boundary condition can be established there. If this is done, it is only necessary to solve the problem half-way across the distance between the slabs, the solution for the other half being symmetrical.

In the same way, if the initial temperature distribution in our homogeneous medium had been antisymmetrical with respect to station 4-1/2, we could have treated the problem for \( N \) cells by setting \( u_0 = u_N/2 = 0 \) and solving the \( N/2 \)-cell problem. Here we must obviously have an even number of cells to begin with, whereas in the symmetrical case we needed an odd number of cells.

It is evident that by considering symmetry effects the number of amplifiers needed may often be cut in half. Furthermore, any arbitrary initial temperature distribution can always be split into a symmetrical and antisymmetrical form. The solution for each of these initial distributions can then be found, and since the equations are linear, the final solution is the sum of the two solutions. Of course this procedure will only work when the conductivity \( K \) and the specific heat capacity \( \rho c \) for the medium are constant or symmetrical about the center of the medium. Also, the boundary conditions must be symmetrical.

**Summary of Investigation of the Use of Difference Techniques for the Heat Equation**

We have shown that it is both simple and straightforward to solve the heat equation with the electronic differential analyzer by replacing spatial derivatives with finite differences. Normal mode shapes show exact agreement with those calculated by separation of variables for the simple problems considered. Decay constants corresponding to the various modes also show good agreement but tend to be somewhat lower than the values calculated by separation of variables, particularly for higher modes or if fewer cells are used. For most engineering problems the order of eight to sixteen cells per spatial dimension should be completely adequate (see Figure 4).

Only one operational amplifier is needed per cell, although in some problems it may be more convenient to use three amplifiers per cell. The problem is completely stable, and the final outputs of the computer are temperature and heat flow as a function of spatial coordinates and time.
One of the most important partial differential equations met in engineering is the wave equation. If we let \( \phi \) represent the magnitude of a disturbance in any medium in which wave propagation can take place, then we can write the wave equation as

\[
\nabla^2 \phi = \frac{1}{v^2} \frac{2\phi}{\partial t^2}
\]

(29)

Here \( v \) is the wave velocity in the medium and \( t \) is the time variable. Equation 29 must of course be subject to spatial boundary conditions and initial time conditions.

The spatial derivatives of the wave equation have exactly the same form as the heat equation, but the time derivative is second order instead of first order. The difference techniques for converting the partial differential equation to system of ordinary differential equations in time are also practically identical. Space does not permit discussion of specific examples, but it may be remarked that application of the method to the problem of a vibrating string has given satisfactory results.

The solution of the wave equation by difference methods is easily performed by the differential analyzer when the medium of propagation is non-uniform. Also, the effect of damping forces can readily be included.

Thus far we have considered partial differential equations with boundary conditions occurring a finite distance apart. It seems evident that our difference techniques as used here are limited to this type of equation. Thus, it would not seem possible to solve problems in semi-infinite or infinite media unless one can let the time variable in the computer represent the spatial variable which goes to infinity.

It should be straightforward to solve problems having spatial coordinate systems other than Cartesian, e.g., cylindrical, spherical, etc. For the appropriate geometries this would undoubtedly require many less cells to realize a desirable accuracy.

**VIBRATING BEAMS**

It would seem of particular engineering interest to investigate the usefulness of the difference technique in solving the problem of flexural vibration of beams. Consider the vibrating beam shown in Figure 5. If we limit ourselves to the transverse deflection shown and assume that the flexural planes remain parallel, then for small deflections the equation of motion is given by
where
\[ x = \text{horizontal distance from the left end of the beam} \]
\[ t = \text{time} \]
\[ y(x,t) = \text{transverse deflection of the beam at any instant} \]
\[ p(x) = \text{mass per unit of beam at } x \]
\[ EI(x) = \text{flexural rigidity at } x \]

The bending moment \( M(x,t) \) is given by
\[
M(x,t) = EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \quad (31)
\]
and the shear is given by
\[
V(x,t) = \frac{\partial}{\partial x} M(x,t). \quad (32)
\]

Equation 30 is of course subject to both boundary and initial conditions. One type of beam of considerable engineering interest is the cantilever beam (one end built-in, the other free). We will also consider the hinged-hinged beam because it is the easiest to analyze theoretically and will give us a good idea of how the other beams will behave when difference techniques are used.

**Derivation of the Difference Equation for the Vibrating Beam**

Once again the partial differential Equation 30 for the vibrating beam is converted into a set of ordinary differential equations by using the difference technique. Thus, distance along the beam is broken into \( N \) segments of width \( \Delta x \); the displacement \( y_n \) at the \( n \)th station will then be a function of time only. We have from Equation 30 as the equation of motion of the \( n \)th cell
\[
(\Delta x)^2 \rho_n \frac{d^2 y_n}{dt^2} + M_{n+1} - 2M_n + M_{n-1} = 0 \quad (33)
\]
where
\[
M_1 = \frac{EI_1}{(\Delta x)^2} (y_{1+1} - 2y_1 + y_{1-1}) \quad (34)
\]

We also note that
\[
V_{n-1/2} = \frac{M_n - M_{n-1}}{\Delta x} \quad (35)
\]
and
\[
\frac{\partial y}{\partial x}_{n-1/2} = \frac{y_n - y_{n-1}}{\Delta x} \quad (36)
\]
Before writing down the complete set of difference equations for \( N \) cells, it is necessary to consider the boundary conditions. Assume we have an \( N \) cell beam and wish to impose the boundary conditions associated with a particular end fastening, e.g., a free end at the right hand extremity of the beam. This means that both the shear \( V \) and bending moment \( M \) must vanish at the beam end. Let us assume, then, that the end occurs at \( N+1/2 \) and that \( V_{N+1/2} = 0 \). From Equation 35 this implies that \( M_{N} = M_{N+1} = 0 \). But from Equation 33 this means that

\[
\frac{d^2}{dt^2} \rho N \frac{d^2 y_n}{dt^2} + M_{n-1} = 0
\]

\[
\frac{d^2}{dt^2} \rho N-1 \frac{d^2 y_{n-1}}{dt^2} = 2M_{n-1} + M_{n-2} = 0.
\]

The remainder of the equations are similar to Equation 33 until the left-hand boundary is reached, at which point the difference equations again depend on the type of end fastening.

Following the same line of reasoning as above, one obtains the following set of conditions for the difference equations for various end fastenings of an \( N \) cell beam:

<table>
<thead>
<tr>
<th>End</th>
<th>Where End Occurs</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free</td>
<td>( N+1/2 )</td>
<td>( M_{N} = M_{N+1} = 0 )</td>
</tr>
<tr>
<td>Hinged</td>
<td>( N )</td>
<td>( M_{N} = y_{N} = 0 )</td>
</tr>
<tr>
<td>Built-in</td>
<td>( N+1/2 )</td>
<td>( y_{N} = y_{N+1} = 0 )</td>
</tr>
</tbody>
</table>

The actual way in which these conditions modify the difference equations is best seen by considering a specific type beam, as in the next section.

**Computer Circuit for Solving the Cantilever Beam by Difference Techniques**

Since it involves both a free end and a built-in end, the cantilever beam shown in Figure 5 seems the best choice for a specific example. The left-hand end of this beam occurs at station \( 1/2 \), while the right-hand end occurs at station \( N+1/2 \). From Equations 33, 34, and 36 along with boundary conditions the computer circuit shown in Figure 6 is obtained. Initial conditions on \( y_n \) and \( y_0 \) must of course be specified in an actual problem. Notice that even though the left-hand end of the beam occurs at station \( 1/2 \), the displacement \( y_1 \) at station \( 1 \) is held fixed at zero.

**Theoretical Solution of the Difference Equations for Vibrating Beams**

In order to check the accuracy of the difference method for beams, we will now solve for the normal modes of vibration of a cantilever beam by separation of variables. When the space and time variables are separated in the equation for
a cantilever beam and the normal-mode frequencies are determined, the following
values are obtained for the dimensionless frequency parameter

\[ \beta_n = \omega_n \sqrt{\frac{EI}{\rho l^4}} \]

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\beta_n & 3.516 & 22.03 & 61.7 & 121.0 & 199.8 \\
\end{array} \]

If the units are selected so that \( \Delta x = 1 \), Equation 37 represents the difference
equation for \( y \) when the cantilever beam is broken into cells.

\[ \frac{\rho}{EI} \frac{d^2 y_1}{dt^2} + y_{1+2} - 4y_{1+1} + 6y_{1} - 4y_{1-1} + y_{1-2} = 0 \]  \( (37) \)

Let us consider 8 cells and let \( \rho/EI = 1 \). When the boundary conditions outlined
previously are applied (left end built in, right end free) a set of seven simulta-
neous difference equations result.

As before, we assume \( y_1 \) varies with time as \( \sin \omega t \). The difference equations
are then reduced to 7 simultaneous algebraic equations. Eliminating the \( y \)'s
gives us

\[ 1 - 336 \lambda^2 + 3312 \lambda^4 - 4140 \lambda^6 + 2432 \lambda^8 - 456 \lambda^{10} + 36 \lambda^{12} - \lambda^{14} = 0 \]  \( (38) \)

The roots of the above polynomial in \( \lambda^2 \) are the normal mode frequencies. The
first four values of \( \lambda_n \) obtained from Equation 38 are

\[ \lambda_1 = 0.0554, \lambda_2 = 0.347, \lambda_3 = 0.940, \lambda_4 = 1.66. \]

For our 8-cell cantilever beam \( \Delta x = 1 \) and \( L = N = 8 \). Also, \( EI/\rho = 1 \) and hence
the dimensionless normal-mode frequency \( \beta_n \) is obtained by multiplying \( \lambda \) by 64.
In the following table \( \beta_n \) for the 8-cell beam is compared with \( \beta_n \) for the con-
tinuous beam.

<table>
<thead>
<tr>
<th>Mode</th>
<th>(continuous beam)</th>
<th>(8 cells)</th>
<th>% deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.516</td>
<td>3.545</td>
<td>+ 0.8</td>
</tr>
<tr>
<td>2</td>
<td>22.03</td>
<td>22.2</td>
<td>+ 0.8</td>
</tr>
<tr>
<td>3</td>
<td>61.7</td>
<td>60.1</td>
<td>- 2.5</td>
</tr>
<tr>
<td>4</td>
<td>121.0</td>
<td>111.3</td>
<td>- 8.0</td>
</tr>
</tbody>
</table>

Evidently an 8-cell uniform cantilever beam (actually requiring only 22 operational
amplifiers) gives tolerable normal-mode frequencies for the first four modes. For
many engineering problems this would be entirely adequate.

In Figure 7 two mode shapes are compared with those for the continuous beam.
Agreement seems to be entirely satisfactory.
Computer Solution for an 8-Cell Uniform Cantilever Beam

The 8-cell cantilever beam discussed in the previous section was set up on the electronic differential analyzer using integrator time constants of 0.2 seconds. Normal-mode frequencies were obtained by driving the cells at anti-nodal points with sinusoidal voltages at the normal-mode frequency. The following table compares the computer and theoretical normal-mode frequencies.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Continuous Beam</th>
<th>8-Cell Theoretical</th>
<th>8-Cell Computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0437 cps</td>
<td>0.442 cps</td>
<td>0.436 cps</td>
</tr>
<tr>
<td>2</td>
<td>0.275</td>
<td>0.276</td>
<td>0.274</td>
</tr>
<tr>
<td>3</td>
<td>0.768</td>
<td>0.750</td>
<td>0.747</td>
</tr>
<tr>
<td>4</td>
<td>1.507</td>
<td>1.39</td>
<td>1.37</td>
</tr>
</tbody>
</table>

The input resistors used in the circuit for computing differences were calibrated to 0.01 per cent, while the resistors representing flexural rigidity EI and mass per unit length \( P \) were calibrated to about 0.5 per cent. It was found that a 1 per cent change in one of the input resistors used for taking differences perturbed in the period of the fundamental normal mode by the order of 1 per cent.

Computer Solution for an 8-Cell Non-Uniform Cantilever Beam

An 8-cell non-uniform cantilever beam was simulated on the differential analyzer. This beam had the configuration shown in Figure 8 and represents an aircraft wing with taper ratios of 2:1 in both chord and thickness. Two problems were solved: (1) the tapered beam alone, and (2) the tapered beam with the addition of a concentrated load at station 8. This load simulated a wing tank of half the weight of the wing. A gust load of one-second duration was applied as a force proportional to the chord at each cell. Figure 9 is a sample of the deflection at station 8, and also shows the one second gust load. No structural damping was included in these examples, but it can be accomplished by simply connecting the appropriate resistors across the integrating condensors.

An integrator time constant of 0.2 seconds was used.

The normal-mode frequencies were obtained for the 8-cell beam with and without the wing tank.

These are tabulated along with the values for normal-mode frequencies obtained by a differential analyzer solution of the eigenvalue problem for the case without wing tank.

<table>
<thead>
<tr>
<th>Mode</th>
<th>8-Cell Eigenvalue (No Wing Tank)</th>
<th>8-Cell Difference (No Wing Tank)</th>
<th>8-Cell Difference (With Wing Tank)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0773 cps</td>
<td>0.0767 cps</td>
<td>0.0372 cps</td>
</tr>
<tr>
<td>2</td>
<td>0.331 cps</td>
<td>0.326 cps</td>
<td>0.256 cps</td>
</tr>
<tr>
<td>3</td>
<td>0.820 cps</td>
<td>0.784 cps</td>
<td>0.709 cps</td>
</tr>
<tr>
<td>4</td>
<td>2.016 cps</td>
<td>2.000 cps</td>
<td>3.279 cps</td>
</tr>
</tbody>
</table>
An interesting structural condition is illustrated in Figure 10, where the moments near the root (station 1) and near mid span (station 4) are recorded for the cases with and without wing tank. For both recordings identical one-second gust loads were applied. The case with wing tank has much lower moments and therefore is not as critical structurally as the case with no wing tank.

Real-Time Simulation of Beam Structures

One of the advantages of electronic differential analyzers over other types of analog computers is the extreme flexibility of their time scales. For example, it is possible with the analyzer to simulate on a one-to-one time scale the aircraft wing discussed in the previous section. If integrator time constants of 0.02 seconds are used, the frequency of the fundamental mode for the 8-cell tapered wing becomes 0.77 cycles per second, which would be the order of magnitude of the frequency for an actual structure.

There is one important effect which must be considered when so speeding up the time scale. Limitations in bandwidth of the dc amplifiers may cause the beam oscillations for the higher modes to exhibit a slight exponential buildup in magnitude. However, the introduction of a small amount of viscous damping (considerably less than actually exists in the structure) easily stabilizes the circuit.

A real-time simulation of structural characteristics could be very useful in testing autopilot designs for some of the larger aircraft and missiles, where structural characteristics must be considered in solving the control problem.

Effect of Small Voltage Transients

The deflection of a beam for a given force is proportional to the fourth power of the length of the beam. Thus as more and more cells are added to the differential-analyzer circuit, the output voltages representing beam deflections becomes more and more sensitive to voltage inputs. The latter may be purposely introduced to simulate forces, or may be inadvertently introduced as a result of power-supply fluctuations or transient voltages when the initial conditions are released. The sensitivity of the network to such disturbances will increase as the fourth power of the number of cells. With electronically regulated power supplies and dc amplifiers which were manually balanced to within 0.01 volt referred to input, no particular difficulty was experienced with the above effect in solving the 8-cell beam problem.

Summary of Difference Technique for Vibrating Beams

The vibrating-beam equation has been solved with the difference method both theoretically and with the electronic differential analyzer. Results show that normal-mode shapes and frequencies exhibit good agreement with continuous beams providing 3 or more cells per half-wave length of the normal mode are used.

Cantilever and hinged-hinged beams have a fixed equilibrium position relative to their surroundings and hence are stable on the electronic differential analyzer. Free-free beams are not supported, however, and will tend to be unstable, since any small voltage unbalance will cause them to rotate and translate as well as vibrate.
Damping can easily be included in the beam equation by placing the appropriate resistors across integrating condensors. Any variable force as a function of time can be introduced at any point or points along the beam. The final computer response gives directly the bending moment and displacement as a function of time and distance along the beam.

This same difference method can be used to solve beams with both torsional and lateral bending. In this case the torsional equation is similar to the wave equation mentioned earlier. The proper cross-coupling resistors then tie the two systems together.

For a more complete discussion of some of the material in this paper, see Reference 6.

DESCRIPTION OF THE DIFFERENTIAL ANALYZER TO BE USED AT THE UNIVERSITY OF MICHIGAN FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

An 80-amplifier differential analyzer is being constructed to solve partial differential equations by the difference method. Forty of the dc amplifiers can be used as integrators or summers; the remaining 40 are summers. The inputs and outputs to the amplifiers are brought out to polystyrene patch panels on the front of the relay racks. Input and feedback impedances and the various computer-circuit connections are patched directly into the polystyrene panels. There are 16 amplifiers per relay-rack, with five racks in all.

The dc amplifiers consist of three stages of triode amplification (2-5691 RCA red tubes) and an optional 12SN7 cathode follower for high-power output operation. The amplifiers are flat to about 40kc with a gain of unity. Plug-in drift stabilizing units using Leeds and Northrup Type STD, 3338-1 Converters hold the dc unbalance referred to input to the order of 100 microvolts.

Using the above equipment it will be possible to solve heat, wave, and beam equations in one spatial dimension with up to 20 cells.

BIBLIOGRAPHY


2. Howe, Carl E., Further Application of the Electronic Differential Analyzer to the Oscillation of Beams, External Memorandum UMM-47 (June 1, 1950), University of Michigan Engineering Research Institute, AF Con, W33-038-ac-14222 (Project MX-794).


Fig. 1 Station Arrangement for N = 10, Heat Equation

Fig. 2 Computer Circuit for Solving the General Heat Equation with Temperature = 0 at x = 0 and Heat Flux = 0 at x = L = (N+1/2)Ax.

Fig. 3 Temperature Distribution as Function of Time
Fig. 1
Percentage Deviation in the Decay Constant as a Function of the Number of Cells.

Fig. 5 Vibrating Beam

Fig. 6
Computer Circuit for Solving the Cantilever Beam by the Difference Method.

Fig. 7
Comparison of Mode Shapes for 8-Cell Cantilever Beam and Continuous Beam.

Fig. 8
Non-Uniform Cantilever Beam with Concentrated Load.
Fig. 9 Sample Solution of 8-Cell Non-Uniform Beam

Fig. 10 Comparison of the Moments at Stations One and Four for the Non-Uniform Cantilever Beam with and without the Concentrated Load
THE NORDSIECK COMPUTER

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I Introduction

After some experience with a mechanical differential analyzer of the Bush type\(^1\), the author became convinced that there was a need for a smaller, cheaper instrument of this type with a faster and more convenient setup procedure. The large instruments are so large and expensive mainly because of the torque amplifiers they contain and are somewhat inconvenient to use because, aside from considerations of accessibility to the individual researcher, they have rather long and complicated setup procedures.

The need for torque amplification can be eliminated by employing a mechanical integrator which will transmit appreciable torque without slipping at all. The author has developed such an integrator, which will be described in section II. Speed and convenience of setup can best be achieved by arranging to have all interconnections made electrically, and accordingly all the input and output variables of all the units (integrators, multipliers, adders, etc.) in the new instrument are converted into electrical form by synchro motors and generators\(^2\) and made available at a plug board. The general size can also be kept small without prejudice to accuracy because the key parts can readily be machined to an accuracy of 0.001 inch and because the ultimate output consists of driving revolution counters or pushing pens across graph paper. These processes require negligible forces and torques, so that if care is taken not to dissipate torque needlessly in bearing friction or otherwise, very small motors can be used throughout. The torques employed in the new machine are in the range of one inch-ounce and the total mechanical driving power is less than \(1/100\) horsepower. The six-integrator machine is about the size of a desk and weighs about 500 lbs. and requires less than 500 watts of 110 volt 60 cycle power. The accuracy cannot be given in any absolute way since it depends on the problem being solved, but it is in the general range of one part in a thousand.

Further reduction in weight and size, for a given number of integrators and with no loss in accuracy, may be possible.

The original machine of this type has been in use at the University of Illinois, Urbana, Illinois for some time and replicas of it have been built and operated at Purdue University, Lafayette, Indiana, and at Radiation Laboratory, University of California, Berkeley, California. A wide variety of problems have been solved on the machines, ranging from stability of non-linear servos and design of non-linear springs to charged particle orbits in linear accelerators and problems in quantum mechanics and nuclear physics.