Aircraft dynamic problems present a major field for the application of modern digital and analog computers. While it is true that airplanes were designed before the advent of these computers, the design requirements of aircraft have advanced simultaneously with the computer development programs. Today the efficient design of high speed aircraft depends to a large extent on the rapidity with which a large volume of calculations on dynamic problems can be completed.

The applications of computers may best be illustrated by considering four general categories of dynamic problems:

1. The problems dealing with long period control of aircraft and missiles during part of or the entire flight time;
2. Rigid body dynamic problems;
3. Problems of the elastic aircraft at zero frequency; and
4. The elastic aircraft with finite inertia, the flutter problem.

Before discussing each category, it is necessary to point out that the four classifications are extremely dependent on each other. It is only because of simplifying assumptions that one is able to consider them separately.

In the first category a missile or airplane is considered to be at rest or in flight. If the craft is at rest, some propulsive force is applied and the resulting motion is to be computed. If the craft is in flight some natural or induced disturbance is applied to the craft and again the resulting motion is to be computed. As an example of problems typical to this group, consider a missile trajectory problem. Assume, for the purpose of simplification, that the missile is constrained to move in only two directions: range and altitude. At any instant of time the geometrical conventions in Figure 1 are used to describe its motion and position in its flight path.

The trajectory of the missile is determined by the following equations:

\[
\ddot{y} = \frac{e}{W} \left[ (T - C)(\cos \theta \cos \alpha - \sin \theta \sin \alpha) - N(\sin \theta \cos \alpha + \cos \theta \sin \alpha) \right]
\]

\[
\ddot{z} = \frac{e}{W} \left[ (T - C)(\sin \theta \cos \alpha + \cos \theta \sin \alpha) + N(\cos \theta \cos \alpha - \sin \theta \sin \alpha) \right] - g
\]
In these equations the parameters are defined as follows: T, the thrust which is directed along the axis of the missile; C, the chord force opposite to T along the same axis; \( \phi \), the climb angle; and \( \alpha \), the angle of attack. \( N \) is the normal force on the missile and \( W \) its weight. The angle \( \alpha \) may vary between 0 and 30 degrees while \( \phi \) may lie between plus or minus 90 degrees, a range which indicates the high degree of non-linearity of the system. As is indicated in Figure 2, some of these quantities are discontinuous functions of time.

The missile is launched with the booster motor burning. At the end of a specified time the booster is dropped and the missile glides prior to motor burning. It is at this point that the calculation of the trajectory begins. On the graphs in Figure 2, time \( t \) is actually the beginning of the glide phase. \( t \) denotes the time of missile motor burning and is accompanied by a corresponding loss in weight due to fuel consumption. At the time of motor burnout, \( t \), the thrust returns to zero and the weight following the jettisoning of excess fuel remains constant.

In the actual integration of the equations of motion a step by step method of solution is employed. At time \( t \), the values of \( \dot{y} \), \( \ddot{y} \), \( \dot{\phi} \), \( \ddot{\phi} \) are available. The corresponding values of these quantities for time \( t + \Delta t \) may be approximated by the following formulas:

\[
\begin{align*}
\dot{y}(t+\Delta t) & = \dot{y} + \ddot{y} \Delta t \\
\ddot{y}(t+\Delta t) & = \ddot{y} + \frac{\Delta \dot{y}}{2} \{ \dot{y} + \ddot{y}(t) \}
\end{align*}
\]

Using these values \( \ddot{y}(t+\Delta t) \) may be formed. Then the values \( \dot{y}(t+1), \ddot{y}(t+1), \dddot{y}(t+1) \) may be improved by the following equations:

\[
\begin{align*}
\ddot{y}(t+1) & = \ddot{y} + \frac{\Delta \ddot{y}}{2} \{ \dot{y} + \ddot{y}(t) \} \\
\dddot{y}(t+1) & = \dddot{y} + \frac{\Delta \dddot{y}}{2} \{ \dot{y} + \ddot{y}(t) \} + \frac{\Delta^2 \dddot{y}}{6} \{ \dddot{y} - \dddot{y}(t) \}
\end{align*}
\]

which may then be used to determine \( \dddot{y}(t+1) \). An identical procedure is used to form \( \dddot{z}(t+1), \dddot{\phi}(t+1), \dddot{\phi}(t+1) \).

This integration process has proved to be quite accurate as is evidenced by the trajectory in Figure 3. The graph records the time history of an actual missile flight together with the trajectory obtained by the procedure outlined above. The actual path of motion is indicated by the solid line, while the digital reproduction of this run, which had for its initial conditions experimental values obtained from the end of boost conditions, is the solid line with circles.

In the second group, problems involving the rigid airplane are studied. The craft is allowed to have 6 degrees of freedom, (Figure 4), with aerodynamic and inertia forces acting upon it. For stability calculations the degrees of freedom may frequently be considered in groups. For instance, the pitching and vertical motion are considered separately from rolling, yawing and sideslip for small amplitudes of motion. The former are the longitudinal degrees of freedom and the latter are the lateral degrees of freedom. Providing linear aerodynamic forces are employed, the solution of the stability problem is relatively simple. The
following equations are typical of the lateral stability problem.

\[
I_x \ddot{\phi} - C_{L_{\phi}} \dot{\phi} - I_x \dot{\gamma} - C_{L_{\gamma}} \dot{\gamma} - C_{L_{\alpha_0}} \beta = 0
\]
\[
I_z \ddot{\gamma} - C_{\alpha} \dot{\gamma} - I_x \ddot{\phi} - C_{\alpha_{\phi}} \dot{\phi} - C_{\alpha_{\alpha_0}} \beta = 0
\]
\[
\frac{2}{\sqrt{\gamma + \beta}} \left[ \dot{\psi} + \beta \right] - C_{\gamma} \dot{\phi} - C_{\gamma_{\phi}} \dot{\phi} - C_{\gamma_{\alpha_0}} \beta = 0
\]

The Army and Navy specifications require that lateral directional oscillation be damped within a certain time, depending on the period of oscillation, Figure 5. The motion of particular concern here is one in which the plane yaws and rolls to the right, followed by a recovery toward the equilibrium position and an overshoot consisting of a yaw and roll to the left. If this motion is not sufficiently damped, it may become very bothersome.

The moments of inertia in the equations may be roughly computed by slide rule using standard formulas, or for more accurate determination computing equipment is frequently employed. The coordinates employed in the above equations are the same as those defined in Figure 4, \( I_x, I_{xz} \), etc. are the moments and products of inertia. The C's are the aerodynamic coefficients. Wherever possible the aerodynamic terms of the stability problem are derived from wind tunnel scale models. If these data are not available, the derivatives are estimated by ratio with other airplanes of similar but known aerodynamic characteristics.

Having determined the coefficients of the equations, an Eigenvalue problem remains to be solved. The real part of the root will yield the damping and from the imaginary part the period is determined. With this information the degree of lateral stability may be located on the specifications chart Figure 5.

Because of the relative sizes of the coefficients in the equations of motion, it is difficult to solve for the roots. This problem is presently being solved by first finding the characteristic equation by direct expansion, and then solving the equations by any one of numerous methods. The expansion and solution is programmed for one continuous operation on the IBM card programmed calculator (CPC). One particularly valuable method of solution known as the Lin and Barstow method, extracts quadratic factors from the equation. The quadratics are solved by the quadratic formula, and the remaining polynomial is treated in a similar fashion. Frequently linear factors are taken from the polynomial as an alternative method.

As a second example dealing with the rigid airplane, a rolling pull out maneuver will be considered. In this problem the airplane pulls out of a dive at constant acceleration. When the airplane becomes horizontal, the ailerons are deflected initiating a roll, and the aerodynamic forces on the vertical surface may exceed the structural limit during the maneuver. The problem is to calculate these loads for different initial accelerations and for different magnitudes of aileron deflection. The equations used to solve this problem are shown below.

\[
\beta = \alpha \rho - \chi + K_{11} \cos \beta + K_{12} \beta + K_{13} \rho + K_{14} \rho^2 + K_{15} \beta^2
\]
\[
\alpha = \gamma - \beta \rho + K_{12} \cos \beta + K_{22} \alpha + K_{23} + K_{24} \alpha^3 + K_{25} \alpha \beta + K_{26} \alpha \rho
\]
\[
\ddot{\beta} = K_{27} \alpha \beta \rho
\]
\[
\dot{\beta} = K_{31} \beta + K_{32} \rho + K_{33} \beta + K_{34} \alpha + K_{35} \beta + K_{36} \rho
\]
\[
\dot{\alpha} = K_{41} \beta + K_{42} \rho + K_{43} \alpha \rho + K_{44} \beta + K_{45} \rho
\]
\[
\dot{\gamma} = K_{51} \beta + K_{52} \rho + K_{53} \beta + K_{54} \alpha + K_{55} \beta + K_{56} \rho
\]
\[ \cos \theta = \pi \cos \phi - q \cos \psi \]
\[ \cos \phi = \rho \cos \psi - \lambda \cos \theta \]
\[ \cos \psi = q \cos \theta - \rho \cos \phi \]

These are the direction cosines of the gravitational vector.

\( \beta \) is the sideslip angle.
\( \alpha \) is the angle of attack.
\( \rho \) is the rolling velocity.
\( \lambda \) is the yawing velocity.
\( \gamma \) is the pitching velocity.

This problem has gained a great deal of complexity from the previous example, since aerodynamic forces are now considered to be non-linear. A complete discussion of the terms in these equations is not consistent with the purpose of this paper. It is sufficient to note that the inertia terms are included in the coefficients \( K_{11}, K_{12}, K_{13}, K_{14}, K_{15} \) and the remaining coefficients define the aerodynamic forces. These coefficients are also obtained from wind tunnel data wherever possible. Since the time history of the velocity of the vertical surface is required, the roots associated with an Eigenvalue problem would not immediately yield the desired information. To solve this problem the REAC was found to yield fast, accurate solutions for various inputs. The REAC diagram for this problem is shown in Figure 6. Blocks A through E integrate the 5 differential equations. In some cases it is necessary to use an isolating amplifier to distribute the output of the integrators to the different parts of the circuit. Blocks F, G and H are used to satisfy the equations determining the gravitational vector. Blocks I, K, L, M, N, P, R, are the servo units producing the non-linear terms. The term denoted by \( K_{3} \) determines the rolling moment due to aileron deflection. This term is represented on the REAC by the relay circuit in the lower left hand corner. The input deflection is a square wave of arbitrary magnitude.

The results are obtained by recording the time history of the sideslip angle \( \beta \). For large values of \( \beta \) the aerodynamic force on the vertical tail surface exceeds the structural limit and failure occurs.

In the problems of the first two categories it is assumed that the aircraft is responding as a rigid body to the imposed forces. In the present section the problems under consideration will contain the actual deformations of the wing under aerodynamic forces. These studies will be carried out with the omission of the inertia terms present in the previous categories. The example problem of this category is that of finding the span loading of a flexible swept wing. It is generally convenient to divide the wing into sections as shown in Figure 7. Each section will be coupled to the adjacent section by springs computed from the stiffness properties of the wing. The springs are arranged to allow the wing to bend in a vertical plane and to twist about its spanwise axis. In addition to the spring restraints, aerodynamic lift and moment act upon each section. Assume that the aerodynamic constants of each section, the design load on the rigid wing, and the built in angle of attack distribution are known. From these conditions the strength engineer can compute the stiffness the wing requires to support the load. As soon as this finite stiffness is introduced, the wing is
free to deform, and this deformation will in turn redistribute the load on the wing. The following iterative procedure is successfully employed to compute the final load and deformation of the wing. The iteration procedure is carried out as one continuous operation on the CPC. With the aerodynamic and assumed elastic parameters and the operating conditions stored in the memory register of the card program calculator, the machine computes a new angle of attack distribution from the rigid body loads by the following formula:

$$\Delta \alpha_n = \frac{1}{G} \left[ \frac{\Delta k}{j} \right]_n \Delta \alpha_n = \frac{\tan \Delta \alpha_n}{E} \left[ \frac{\Delta k}{j} \right]_n M_n$$

A constant angle of attack determined by

$$\alpha_i = - \sum_n \frac{C_n}{S} \Delta \alpha_n$$

is added to the new angle of attack distribution, thus keeping the total load on the wing constant. With this normalized angle of attack distribution, new loads are computed for each section by the following formulas:

$$V_n = \begin{cases} \theta & n = K \\ V_{n+2} + \beta \left( \frac{\frac{\partial}{\partial \alpha} \Delta \alpha \cdot C \right)_n, & \alpha_n, \ n = 2, 4, \ldots K-2 \end{cases}$$

$$M_n = \begin{cases} \gamma & n = K \\ M_{n+2} + \left( \frac{\frac{\partial}{\partial \alpha} \Delta \alpha \cdot C \right)_n, & \alpha_n, \ n = 2, 4, \ldots K-2 \end{cases}$$

With the new wing loading the process may be repeated until convergence is reached. If the loads or angle of attack distribution differ radically from the original loads, the wing stiffness may have to be modified to obtain the desired load and deformation characteristics.

In dealing with the dynamic stability of the airplane in the higher frequency ranges of the fourth category, it is necessary to include not only the aerodynamic forces and elastic deflections, but also the local mass and inertia effects. This
gives rise to very complicated sets of equations, the solution of which leads to a serious computing problem. However, the danger involved in high frequency dynamic instability, generally called flutter, is such that these calculations must be performed, and a great deal of effort has been expended by engineers and mathematicians in trying to find ways of employing computing machines in the solution of this problem.

Two quite different approaches have been used successfully in the Douglas Aircraft Company, and will be described here. The first and older technique is digital in character and uses IBM computing equipment. The second method employs the California Institute of Technology analog computer which sets up electric circuits having the same dynamic characteristics as the aircraft system.

When digital methods are used the work is done almost entirely by matrix manipulation. Although the equations of motion are linear, the aerodynamic forces contain time lags which in the usual formulation lead to complex coefficients in the equations of motion. When the number of degrees of freedom is high (say greater than seven) considerable difficulty is experienced in obtaining accurate values of the frequencies and stability (damping) of all of the possible flutter modes. As a consequence every effort is made to minimize the number of degrees of freedom. The method usually followed uses the natural modes of vibration as degrees of freedom. In other words, it is assumed that the deflection configuration of the airplane structure when flutter occurs can be made up of a linear combination of the natural ground vibration modes. These ground vibration modes may be either calculated or measured during ground vibration tests.

Given these data the detailed procedure is as follows:

The wing or tail surfaces are divided into sections as shown in Figure 7 and the mass, center of gravity, and pitching inertias are computed for each section. Aerodynamic force and moment coefficients must be calculated for each section for a range of ratios of velocity to frequency. When this is done, the final equations may be formed by a process based on the principle of virtual work.

This process leads to the formation of generalized force coefficients for the inertia, aerodynamic, and elastic terms from the sectional values of these quantities. Thus if \( W_i \) represents a mass matrix for the \( i \)-th section and \( N_i \) is the corresponding aerodynamic matrix, then \( \bar{N} = Z_i \phi_i \ast W_i \phi_i \) and \( Z = Z_i \phi_i \ast W_i \phi_i \) will represent generalized force coefficient matrices. The transformation matrix \( \phi_i \) represents the relative deflections at section "i" due to the various modes of vibration employed, and \( \phi_i \ast \) is the transpose of the matrix \( \phi_i \). The generalized elastic constant matrix \( Z \) is usually computed from \( \bar{N} \) and a knowledge of the ground natural frequencies.

If \( j \) is a column matrix of the magnitudes of the various ground modes present in flutter, the final equations become \( (A - Z I)j = 0 \), where \( A = \bar{N} \) and \( Z = \frac{1}{\omega^2} (\phi_i \ast \phi_i) \), \( \omega \) is the frequency of flutter, and \( j \) is an index of the system stability.

The matrix equation \( (A - Z I)j = 0 \) will have non-trivial solutions only if the determinant \( |A - Z I| = 0 \) which yields an algebraic equation in \( \omega \) called the characteristic equation. A number of methods for obtaining the complex characteristic equation have been used. However, in general, a direct expansion of the determinant is used for four degrees of freedom or less. This work
including the computation of the roots of the complex fourth order equation is programmed for one continuous operation on the CPC. For higher orders than the fourth, Leverrier's method is used. This involves computing powers of the matrix \( A = \mathbb{C}^{-1} \mathbb{C}^4 \) and applying the following formulas:

\[
\begin{align*}
S_1 &= \sum \text{Diagonal Terms in Matrix } A \\
S_2 &= \sum \text{Diagonal Terms in Matrix } A^2 \\
S_3 &= \sum \text{Diagonal Terms in Matrix } A^3 \\
S_4 &= \sum \text{Diagonal Terms in Matrix } A^4 \\
S_N &= \sum \text{Diagonal Terms in Matrix } A^N \\
\end{align*}
\]

The characteristic equation is:

\[
\zeta^N + b\zeta^{N-1} + c\zeta^{N-2} + d\zeta^{N-3} + e\zeta^{N-4} \ldots \ldots f = 0
\]

where

\[
\begin{align*}
b &= -1 \ (S_1) \\
c &= -1/2 \ (S_2 + b \ S_1) \\
d &= -1/3 \ (S_3 + b \ S_2 + c \ S_1) \\
e &= -1/4 \ (S_4 + b \ S_3 + c \ S_2 + d \ S_1) \\
f &= -1/N \ (S_N + b \ S_{N-1} + c \ S_{N-2} + d \ S_{N-3} + e \ S_{N-4} \ldots) \\
\end{align*}
\]

In some instances it is found that Leverrier's Method for calculation of the characteristic equation requires that a large number of figures be carried. If sufficient figures are not retained, the last few coefficients of the characteristic equation become inaccurate.

An alternate method for calculation of the last coefficients by determinants is as follows:

Coefficients of a six-degree characteristic equation:

\[
\begin{align*}
e &= \frac{1}{2}(D_{+1} + D_{-1}) - 1 - c - D_0 \\
f &= \frac{1}{2}(D_{+1} - D_{-1}) - b - d \\
g &= D_0 \\
\end{align*}
\]

Coefficients of a seven-degree characteristic equation:

\[
\begin{align*}
f &= -\frac{1}{2}(D_{+1} + D_{-1}) - b - d + D_0 \\
g &= \frac{1}{2}(D_{-1} - D_{+1}) - 1 - c - e \\
h &= -D_0 \\
\end{align*}
\]
In the above equations

\[
\begin{align*}
&b) \quad \text{Coefficients of the characteristic equation previously determined by Leverrier's Method} \\
&c) \quad A = \text{Normalized matrix for which characteristic equation is being determined.} \\
&d) \quad D_0 = A \\
&e) \quad D_{+1} = A - I \\
&f) \quad D_{-1} = A + I
\end{align*}
\]

Having obtained the complex characteristic equation, the roots are found by an iterative process, employing synthetic division and Newton's method.

It is hoped that the foregoing example will sufficiently illustrate the complexity of the flutter problem, and make clear the need for automatic computing machinery in this field.

The other method successfully used in attacking the flutter problem employs the analog computer developed at the California Institute of Technology. In using this machine the wing is also broken into sections, and the same physical properties as before must be computed. An electric circuit is formed which matches each of these sections and their elastic connections. In the electric circuit analogy voltage corresponds to velocity, current to force, capacity to mass and so forth. The aerodynamic forces must be simulated by means of amplifiers. The circuit for a typical section is given in Figures 9A, 88 and 8C.

When the analogical system has been set up, its behavior under various electrical impulses may be observed on an oscilloscope and photographed if desired. The excitation may correspond to the airplane entering a sudden gust, or it might simulate a vibrator placed in a wing. In any case the dynamic stability and frequency of the system may be observed. Examples are given in Figure 9.

One great advantage of the analog method of solution is the ease with which physical parameters of the airplane can be changed. If the digital technique first described is employed and it is desired, for example, to change the wing rigidity, new natural modes of vibration must be calculated before the process described can even begin. On the analog machine only the values of a few inductors need be altered.

In conclusion it may be stated that the Douglas Aircraft Company has successfully made use of a number of different computing machines of both the digital and analog varieties. No comparison of the relative merits of these various types will be made, since experience has shown that each type has advantages depending on the nature of the problem. In any event the use of computing machinery has become practically essential to modern aircraft engineering, and this trend will undoubtedly continue as further developments in this field take place.
Figure 1  Geometry of Missile Flight.

Figure 2  Discontinuous Functions of Missile Flight.

Figure 3  Missile Trajectory

Figure 4  Airplane Degrees of Freedom

Figure 5  Stability Chart.
GENERATION OF TERMS NON-LINEAR IN $\alpha$

Figure 6 (Cont'd.)

GENERATION OF TERMS NON-LINEAR IN $\beta$

BLOCK L

GENERATION OF TERMS NON-LINEAR IN $\beta$

BLOCK M

GENERATION OF TERMS NON-LINEAR IN $\phi$

BLOCK N

GENERATION OF TERMS NON-LINEAR IN $\phi$

BLOCK P

GENERATION OF TERMS NON-LINEAR IN $\phi$

BLOCK Q

GENERATION OF TERMS NON-LINEAR IN $\phi$

BLOCK R

From the collection of the Computer History Museum (www.computerhistory.org)
Figure 7  Section Division of Wing

Figure 8A Analog Circuit for One Flutter Section.

Figure 8B Aerodynamic Force Circuit

Figure 9  Response Curves of Analog Solutions.
A brief review is given of the memory properties of non-linear ferroelectric materials in terms of the direction of polarization.

A sensitive pulse method has been developed for obtaining static remanent polarization data of ferroelectric materials. This method has been applied to study the effect of pulse duration and amplitude and decay of polarization on ferroelectric ceramic materials with fairly high crystalline orientation.

These studies indicate that ferroelectric memory devices can be operated in the megacycle ranges.

Attempts have been made to develop electrostatically induced memory devices using ferroelectric substances as a medium for storing information. As an illustration, a ferroelectric memory using a new type of switching matrix is presented having a selection ratio 50 or more.