1. Introduction

Auctions have been studied in economics and game theory for a long time as important resource allocation mechanisms in distributed environments. In recent years, their role has grown with the emergence of Internet and electronic commerce, as businesses and corporations leverage the new medium to streamline their procurement process. Many businesses are moving to an auction-based purchase method where they issue a request for quotes for the goods and services needed, and let the suppliers bid for a piece of the business. Driven by these fundamentals, auctions and algorithms related to them have become important and popular research topics in computer science.

An exchange generalizes the auction mechanism to the setting with multiple buyers and sellers. Some familiar examples are the exchanges for equities and commodities, transportation, electricity, and the business-to-business exchanges. A combinatorial exchange is an exchange where buyers and sellers can bid on bundles (subsets) of goods. Combinatorial markets are desirable because items often have complementarity, and combinatorial bidding minimizes bidders’ risk of getting stuck with only a partial subset. It also improves the overall economic efficiency.

2. Partial Acceptance of Bids

Most research on combinatorial markets [1, 3, 4] has focused on the binary case where each bid must be either fully accepted or rejected, which makes the problem NP-Complete [2] (and inapproximable [4]). In many real combinatorial markets, bids could be accepted partially, but it is more desirable to accept them entirely or not at all. For instance, transportation carriers prefer full truck loads but will carry partial truck-loads. However, since a combinatorial market maker’s reputation is linked to its ability to allocate full bundles, there is an incentive to minimize the number of partial bids. Allowing bids to be accepted partially has two advantages. First, a better economic allocation is obtained. Second, the computational complexity might be reduced significantly.

In this paper, we study markets where bids can be accepted partially, but it is desirable to only accept a small number of them partially. We show how many bids have to be accepted partially to obtain a solution of equal value as that where all bids can be accepted partially. We also determine the computational complexity of finding such a solution.

3. Clearing Objectives

We consider two natural objective functions in solving combinatorial exchanges: surplus and trade volume. The former is the net monetary gain (profit) realized by trading the goods: the difference between the revenue collected from the buyers and the amount paid to the sellers. Maximizing the surplus is also equivalent to maximizing the social welfare in an economy because it puts the goods in the hands of the people who value them most. The second objective function is the total number of units traded over all goods without external subsidy. That is, the goal is to maximize the trade volume under the constraint that the net surplus is non-negative.

4. Combinatorial Exchanges

Consider an exchange, with \( n \) participants. A seller places a sell bid specifying the goods he wishes to sell and at what price. Similarly, a buyer places a buy bid specifying the goods he wishes to buy and at what price. Consider the optimal solutions corresponding to the two clearing objectives, allowing any number of partial bids. Though it seems that these solutions might have a rather large number of partial bids, we show that there
exist optimal solutions such that the number of partials is bounded by the number of items and is independent of the number of participants. We also show how these solutions can be computed using Linear Programming (LP). We present fast combinatorial algorithms for the special case of single item exchanges.

4.1. Multi-Item Exchanges

We consider multi-item exchanges with \( k \) items. To compute the optimal solutions, we model them as integer programs (IP) and solve the corresponding LP relaxations. The LP corresponding to surplus-maximizing exchange has \( n \) binary variables (one for each bid) and \( k \) equality constraints (one for each item). Since the IP has only \( k \) constraints, the solution of corresponding LP relaxation has at most \( k \) fractional variables, which corresponds to the partially satisfied bids. The IP for trade-maximizing exchange has one additional constraint corresponding to non negative surplus. Therefore, the solution of corresponding LP relaxation has at most \( k + 1 \) partials.

**Theorem 4.1** Consider a multi-item multi-unit combinatorial exchange with \( k \) item types. Given any set of \( n \) combinatorial buy or sell bids, there is a surplus-maximizing matching with at most \( k \) partially satisfied bids, and a trade-maximizing matching with at most \( k + 1 \) partially satisfied bids. These matchings can be found by solving a LP with \( n \) variables and \( k \) constraints.

4.2. Single-Item Exchanges

For the special case of single item exchanges, we can derive fast combinatorial algorithms which compute the optimal solutions in \( O(n \log n) \) time.

Define \( q_i \) to be the number of units participant \( i \) wishes to trade and let \( p_i \) be the per unit price. For surplus-maximizing market, we consider the sellers in ascending and buyers in descending order of per unit price. Let \( i \) and \( j \) be the current seller and buyer respectively. Seller \( i \) trades with buyer \( j \) if \( p_i \leq p_j \) otherwise no additional trade is possible and so algorithm terminates. If \( q_i \leq q_j \), we update \( q_j \) to be \( q_j - q_i \) and consider the next seller else we update \( q_i \) to be \( q_i - q_j \) and consider the next buyer. It is easy to see that in the end at most one bid remains partial.

The algorithm for trade-maximizing market consists of two steps. The first step is similar to the surplus-maximizing algorithm with two differences. First, the sellers and buyers are considered in ascending order of per unit price and second, if \( p_i > p_j \) then we consider the next buyer. It is easy to see that after first step at most one sell bid is partial but number of partial buy bids can be more than one. In the second step, we rearrange the trades such that at most one buy bid remains partial. The second step uses the observation that if \( l \) and \( m \) are two buyers with partially satisfied bids, such that \( p_l > p_m \), then \( l \) can also trade with any seller \( m \) is trading with.

**Theorem 4.2** Consider a single item, multi-unit combinatorial exchange. Given any set of \( n \) combinatorial buy or sell bids, there is a surplus-maximizing matching with at most one partially satisfied bids, and a trade-maximizing matching with at most two partially satisfied bids, which can be determined in time \( O(n \log n) \).

4.3. Exchanges with XOR-Bids

Next we consider the case of xor-bids [4] which are more expressive because they can encode substitutability among items and decreasing marginal utility in units. They are also used in practice for expressing volume discounts. A bid has the form \( (q_1, p_1) \oplus (q_2, p_2) \oplus \ldots \oplus (q_k, p_k) \), where at most one \( (q_i, p_i) \) is to be (partially) accepted. We now ask: is there a polynomial time algorithm for maximizing the surplus or trade volume with xor-bids where the number of partial bids is \( l \). Surprisingly, the answer is negative.

**Theorem 4.3** For exchanges with xor-bids, computing the optimal surplus or trade volume is NP-Complete, even for single-item exchanges (actually even auctions), irrespective of how many partial bids are allowed.

5. Conclusion

In the paper, we have shown a sharp division between markets with superadditive and subadditive valuation functions. The markets where valuation functions are superadditive can be optimally solved using a small number of partial bids (depending on the number of distinct item types). On other hand, if the valuation functions are subadditive, then even accepting bids partially does not help clear the market.

References


