On Closure Operators in Fuzzy Deductive Systems and Fuzzy Algebras *

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Abstract

The starting point of this paper is the classical well-known theorem due to G. BIRKHOFF, P. HALL, *and* J. SCHMIDT *which establishes a one-to-one correspondence between compact closure operators, inductive closure operators, inductive closure systems, and closure operators generated in deductive systems (and generated in universal algebras, respectively). In the paper presented we make first steps in order to generalize this important theorem to the fuzzy set theory and fuzzy deductive systems (and fuzzy algebras, respectively).*

Keywords Compact Closure Operators, Inductive Closure Operators, Inductive Closure Systems, Universal Algebras, Deductive Systems, Fuzzification

1 Introduction

Let $\mathbb{N}$ be the set of all non-negative integers, i. e. $\mathbb{N} = \{0, 1, \cdots\}$. For arbitrary crisp sets A and B by $A \cap B$, $A \cup B$, and $A \setminus B$ we denote the usual intersection, union, and difference of A and B, respectively, furthermore $A \subseteqq B$ means that A is a subset of B. For an arbitrary system $\mathfrak{A}$ of sets by $\bigcap \mathfrak{A}$ and $\bigcup \mathfrak{A}$ we denote the intersection and the union of all sets of $\mathfrak{A}$, respectively. If we have a family $(A_i \,|\, i \in I)$ of sets A_i, then we write $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$. The cardinal number of A is denoted by $\mathrm{card}\, A$, the power set of A by $\mathbb{P}(A)$, the empty set by $\emptyset$, and the empty sequence of elements of a set by e. Hence we define $A^0 =_{\text{def}} \{e\}$. $A^n =_{\text{def}}$ the set of all sequences of elements of A with the length n where n is an integer with $n \geqq 1$. Finally, we define $A^* =_{\text{def}} \bigcup_{n \in \mathbb{N}} A^n$.

For compact denotation in the following we shall use sometimes the symbolic of predicate calculus, i. e. $\forall x$ as "for every x", $\exists x$ as "there is an x", $\wedge$ as "and", $\vee$ as "or", $\rightarrow$ as "if - then", $\leftrightarrow$ as "if and only if", $\neg$ as "not".

Remember the definition of a complete lattice

$$\mathfrak{L} = [L, \curlywedge, \curlyvee, 0, u]$$

with the domain L, the intersection operator $\curlywedge$, the union operator $\curlyvee$, the zero element 0, and the unit element u.

Remember that by the definition

$$x \preceqq y =_{\text{def}} x \curlywedge y = x \qquad (x, y \in L)$$

a partial order on L is introduced. For $K \subseteqq L$ by $\inf K$ and $\sup K$ we denote the infimum and the supremum of K with respect to $\preceqq$, respectively. A set $C \subseteqq L$ is said to be a $\preceqq$-chain of $\mathfrak{L}$ if and only if

$$\forall x \forall y \left(x, y \in C \rightarrow x \preceqq y \quad \vee \quad y \preceqq x \right) .$$

Let $\leqq$ be the natural ordering of real numbers. For an arbitrary set S of real numbers by $\mathrm{Inf}\, S$ and $\mathrm{Sup}\, S$ we denote the infimum and the supremum of S with respect to $\leqq$, respectively. By $\langle 0, 1 \rangle$ we denote the set of all real numbers r with $0 \leqq r \leqq 1$.

Let U be an arbitrary non-empty set called universe. A fuzzy set F on U is a mapping $F : U \rightarrow \langle 0, 1 \rangle$, i. e. we do not distinguish between a fuzzy set F and its membership function μ_F. The set of all fuzzy sets on U is denoted by $F\mathbb{P}(U)$.

We introduce the empty fuzzy set $\not{\!O}$ on U and the universal fuzzy set $\not{U}$ on U, respectively, for every $x \in U$ defined by

$$\begin{aligned} & \not{\!O}(x) = 0 \\ \text{and} \quad & \not{U}(x) = 1 . \end{aligned}$$

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As usual we define the support $\operatorname{supp}(F)$ of a fuzzy set F on U by

$$\operatorname{supp}(F) = \{x \mid x \in U \wedge F(x) > 0\} \ .$$

A fuzzy set F is said to be finite and a singleton if and only if $\operatorname{supp}(F)$ is finite and $\operatorname{card}\operatorname{supp}(F) = 1$, respectively.

For $F, G \in F\mathbb{P}(U)$ as usual we put

$$F \sqsubseteq G =_{\text{def}} \forall x \left(x \in U \rightarrow F(x) \leqq G(x)\right)$$

and for $x \in U$ we define

$$(F \sqcap G)(x) =_{\text{def}} \min\left(F(x), G(x)\right)$$
$$(F \sqcup G)(x) =_{\text{def}} \max\left(F(x), G(x)\right) \ .$$

Furthermore, for arbitrary $\mathfrak{F} \subseteqq F\mathbb{P}(U), x \in U$ we put

$$(\mathrm{INF}\,\mathfrak{F})(x) =_{\text{def}} \operatorname{Inf}\{F(x) \mid F \in \mathfrak{F}\}$$
$$(\mathrm{SUP}\,\mathfrak{F})(x) =_{\text{def}} \operatorname{Sup}\{F(x) \mid F \in \mathfrak{F}\} \ .$$

Remark
The operations $\sqcap, \sqcup, \mathrm{INF}, \mathrm{SUP}$ are defined in the "standard sense" and used throughout the paper presented. In a forthcoming paper we shall discuss the case if $\min$ and $\max$ are replaced by an arbitrary t-norm τ and an s-Norm (t-conorm) σ, respectively. Furthermore, INF and SUP should be replaced by the quantifier Q_τ and Q_σ, respectively (see [19–22]).

Finally, remember the following

Theorem 1
$\mathbb{L} = [\mathbb{P}(U), \cap, \cup, \emptyset, U]$ and $\mathcal{L} = [F\mathbb{P}(U), \sqcap, \sqcup, \not{\mathcal{O}}, \not{\mathcal{U}}]$ are complete lattices.

2 On Compactness of Classical Closure Operators. Algebraic Closure Operators. The Theorem of G. Birkhoff, P. Hall, and J. Schmidt.

Let $\mathfrak{L} = [L, \curlywedge, \curlyvee, 0, u]$ be a complete lattice. Assume that $\varphi : L \rightarrow L$ and $C \subseteqq L$.

Definition 1

1. *φ is said to be* embedding *on* $\mathfrak{L}$ $=_{\text{def}} \forall x (x \in L \rightarrow x \trianglelefteq \varphi(x))$
2. *φ is said to be* closed *on* $\mathfrak{L}$ $=_{\text{def}} \forall x (x \in L \rightarrow \varphi(\varphi(x)) \trianglelefteq \varphi(x))$
3. *φ is said to be* monotone *on* $\mathfrak{L}$ $=_{\text{def}} \forall x \forall y (x, y \in L \wedge x \trianglelefteq y \rightarrow \varphi(x) \trianglelefteq \varphi(y))$
4. *φ is said a* closure operator *of* $\mathfrak{L} =_{\text{def}} \varphi$ *fulfils the conditions 1, 2, and 3.*

Definition 2
C is said to be a closure system of $\mathfrak{L}$
$=_{\text{def}} \forall D \left(D \subseteqq C \rightarrow \inf D \in C\right)$.

We continue with the formulation of the fundamentel theorem due to G. Birkhoff, P. Hall, and J. Schmidt which is important in many branches of algebra and which gives an algebraic characterization of the (classical) monotonic reasoning (see [12, 13], also [7]). In particular, by this theorem the range of applicability of Zorns lemma is well defined (see [12]). (See also the remarks on next page.)

In order to formulate this theorem, we fix an arbitrary non-empty set U and consider the complete lattice

$$\mathbb{L} = [\mathbb{P}(U), \cap, \cup, \emptyset, U] \ .$$

For a mapping $\Phi : \mathbb{P}(U) \rightarrow \mathbb{P}(U)$ and a system $\mathfrak{C} \subseteqq \mathbb{P}(U)$ of subsets of U we define the following well-known fundamental concepts:

Definition 3

1. *Φ is said to be* compact *on* $\mathbb{L}$
$$=_{\text{def}} \forall X \forall y \begin{pmatrix} X \subseteqq U \wedge y \in \Phi(X) \rightarrow \\ \exists X_{fin} \begin{pmatrix} X_{fin} \subseteqq X \wedge X_{fin} \text{ is finite} \\ \wedge y \in \Phi(X_{fin}) \end{pmatrix} \end{pmatrix}$$
2. *Φ is said to be* inductive *on* $\mathbb{L}$
$$=_{\text{def}} \forall \mathfrak{K} \begin{pmatrix} \mathfrak{K} \subseteqq \mathbb{P}(U) \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \text{ is a } \subseteqq\text{-chain} \\ \rightarrow \Phi(\bigcup \mathfrak{K}) \subseteqq \bigcup \{\Phi(K) \mid K \in \mathfrak{K}\} \end{pmatrix}$$
3. *$\mathfrak{C}$ is said to be inductive on $\mathbb{L}$*
$$=_{\text{def}} \forall \mathfrak{K} \begin{pmatrix} \mathfrak{K} \subseteqq \mathfrak{C} \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \text{ is a } \subseteqq\text{-chain} \\ \rightarrow \bigcup \mathfrak{K} \in \mathfrak{C} \end{pmatrix}$$

In order to describe the generation of closure operators by deductive systems, we introduce the following notions. Let n be an integer with $n \geqq 0$ and assume that $X \subseteqq U$.

For many applications, in particular in logic, it is convenient to generalize the notion of *determininistic, total* finitary operation to the concept of finitary *non-deterministic, partial* operation (see [12]). With respect to applications in logic, we prefer the term *deduction rule* in this case.

Definition 4

1. *d is said to be an n-ary* deduction rule *on U if and only if*
$$d \subseteqq U^n \times U \qquad (n \geqq 0) \ .$$
If n is not specified, then d is called a finitary *deduction rule on U.*

2. *$\vartheta = [U, D]$ is said to be a* deductive system *on U if and only if D is a set of finitary deduction rules on U.*

3. *For given $X \subseteqq U$ we put*

$$\overline{D}(X) =_{\text{def}} \left\{ y \,\middle|\, \begin{array}{l} \text{there are a natural number } n \geqq 0, \\ \text{an } n\text{-ary deduction rule } d \in D, \\ \text{and } x_1, \ldots, x_n \in X \\ \text{such that } [x_1, \ldots, x_n; y] \in d \end{array} \right\}.$$

4. *$X \subseteqq U$ is said to be D-*closed *if and only if $\overline{D}(X) \subseteqq X$.*

5. *$\mathfrak{C}_\Phi =_{\text{def}} \{C \,|\, C \subseteqq U \wedge \Phi(C) \subseteqq C\}$*

6. *$\Phi_D(X) =_{\text{def}} \bigcap \left\{C \,\middle|\, X \subseteqq C \subseteqq U \wedge \overline{D}(C) \subseteqq C\right\}$*

Now, we are able to formulate the theorem due to G. BIRKHOFF, P. HALL, and J. SCHMIDT:

Theorem 2
If Φ is a closure operator on $\mathbb{L}$, then the following propositions 1, 2, 3, and 4 are pairwise equivalent:

1. *Φ is compact on $\mathbb{L}$.*

2. *Φ is inductive on $\mathbb{L}$.*

3. *$\mathfrak{C}_\Phi$ is inductive on $\mathbb{L}$.*

4. *there exists a set D of finitary deduction rules on U such that $\Phi = \Phi_D$.*

Remarks

1. Concerning history, we want to mention that the role of compactness in studying logical consequence operators was discovered and investigated by A. TARSKI in [14–16] and emphasized by K. SCHRÖTER in [13]. A. MALCEV introduced the model theoretic version of compactness and established its applicability in several branches of algebra (see [11]).

2. If D is a set of finitary total operations on U, then the equivalence of 3 and 4 was first proved by J. SCHMIDT in [12], but according to P. COHN in [7], page 81, is an unpublished result of P. HALL, and probably G. BIRKHOFF knew this result (at least part of it) even earlier (see [1–4]).

3. The equivalence of 1 and 3 was also discovered by J. SCHMIDT and first proved in [12]. He also pointed out the importance of inductive systems of sets for applying ZORN's lemma, where it should be mentioned that the notion of inductiveness can already be found in [6].

4. A proof for the equivalence of 3 and 4 can also be found in [12], with the only difference that the term "finitary deduction rule" is not used there.

5. The equivalence of 2 and 3 is added in this paper. A proof can be carried out easily by using the methods developed in [12].

6. Using the concept of *Clone* (see [7], for instance) we can prove the following modification of theorem 2: In assertion 4 of theorem 2 D can be taken to be a clone of deduction rules, where a clone of deduction rules is defined in the same way as a clone of operations. If we intend to construct a clone of deduction rules by a given compact closure operator Φ, we shall see that the existence of the projection operations follows from the reflexivity of Φ, whereas the monotonicity and the closedness of Φ together imply that the system of deduction rules is closed with respect to compositions.

3 On Closure Operators generated in Fuzzy Deductive Systems and in Fuzzy Algebras

Now, we intend to "fuzzify" some concepts and results, leading to the classical theorem of G. BIRKHOFF, P. HALL, and J. SCHMIDT.

For formulating the following definitions we assume that $\Psi : F\mathbb{P}(U) \to F\mathbb{P}(U)$, $\Gamma \subseteqq F\mathbb{P}(U)$.

Definition 5
Ψ is said to be a closure operator on $\mathcal{L}$
If 1. $\forall F \left(F \in F\mathbb{P}(U) \to F \sqsubseteq \Psi(F)\right)$

2. $\forall F \left(F \in F\mathbb{P}(U) \to \Psi\left(\Psi(F)\right) \sqsubseteq \Psi(F)\right)$

3. $\forall F \forall G \left(\begin{array}{c} F, G \in F\mathbb{P}(U) \wedge F \sqsubseteq G \\ \to \Psi(F) \sqsubseteq \Psi(G) \end{array} \right)$

Definition 6

1. *Ψ is said to be* compact *on $\mathcal{L}$*

$$=_{\text{def}} \forall F \forall y \left(\begin{array}{l} F \in F\mathbb{P}(U) \wedge y \in U \to \\ \exists F_{fin} \left(\begin{array}{l} F_{fin} \in F\mathbb{P}(U) \\ \quad \wedge F_{fin} \sqsubseteq F \\ \quad \wedge F_{fin} \text{ is finite} \\ \wedge \Psi(F)(y) \leqq \Psi(F_{fin})(y) \end{array} \right) \end{array} \right)$$

2. *Ψ is said to be* inductive *on $\mathcal{L}$*

$$=_{\text{def}} \forall \mathfrak{K} \left(\begin{array}{l} \mathfrak{K} \subseteqq F\mathbb{P}(U) \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \text{ is a } \sqsubseteq\text{-chain} \\ \to \Psi(\mathrm{SUP}\, \mathfrak{K}) \sqsubseteq \mathrm{SUP} \left\{\Psi(K) \,\middle|\, K \in \mathfrak{K}\right\} \end{array} \right)$$

3. *Γ is said to be* inductive *on $\mathcal{L}$*

$$=_{\text{def}} \forall \mathfrak{K} \left(\begin{array}{r} \mathfrak{K} \subseteqq \Gamma \wedge \mathfrak{K} \neq 0 \wedge \mathfrak{K} \text{ is a } \sqsubseteq\text{-chain} \\ \to \mathrm{SUP}\, \mathfrak{K} \in \Gamma \end{array} \right)$$

Because every n-ary fuzzy operation can be considered as a special case of an n-ary fuzzy deduction rule, we formulate the following definitions, lemmata, and theorems only for finitary fuzzy deduction rules.

Definition 7

1. *δ is said to be an n-ary* fuzzy deduction rule *on U*

$$=_{\text{def}} \delta : U^n \times U \to \langle 0,1\rangle \ .$$

2. *If n is not specified, we will speak of a finitary fuzzy deduction rule on U. If $x_1, \ldots, x_n, y \in U$, then we interpret the real number*

$$\delta(x_1, \ldots, x_n, y)$$

as the logical value that the fuzzy deduction rule δ has the output y for the inputs $x_1, \ldots, x_n$.

3. *$\vartheta = [U, \Delta]$ is said to be a* fuzzy deductive system *on U $=_{\text{def}}$ Δ is a set of finitary fuzzy deduction rules on U.*

4. *For a given fuzzy set $F \in F\mathbb{P}(U)$ we put*

$$\overline{\Delta}(F)(y) =_{\text{def}} \operatorname{Sup}\left\{ \min\begin{pmatrix} F(x_1), \ldots, F(x_n), \\ \delta(x_1, \ldots, x_n; y)\end{pmatrix} \middle| \begin{array}{r} n \in \mathbb{N} \wedge x_1, \ldots, x_n; \in U \\ \wedge\, \delta \in \Delta \wedge \delta \text{ is } n\text{-ary}\end{array}\right\}$$

5. *F is said to be* Δ-closed $=_{\text{def}} \overline{\Delta}(F) \sqsubseteq F$

6. $\Psi_\Delta(F) =_{\text{def}} \operatorname{INF}\left\{ G \middle| \begin{array}{r} G \in F\mathbb{P}(U) \wedge F \sqsubseteq G \\ \wedge\, \overline{\Delta}(G) \sqsubseteq G\end{array}\right\}$

Lemma 3

The mapping

$$\overline{\Delta} : F\mathbb{P}(U) \to F\mathbb{P}(U)$$

is monotone on $\mathcal{L}$.

Theorem 4

1. *The set $\left\{F \middle| F \in F\mathbb{P}(U) \wedge \overline{\Delta}(F) \sqsubseteq F\right\}$ of all Δ-closed fuzzy sets F on U is a closure system of the lattice $\mathcal{L}$.*

2. *Ψ_Δ is a closure operator of the lattice $\mathcal{L}$.*

Definition 8

1. $\Delta^{[0]}(F) =_{\text{def}} F$

2. $\Delta^{[k+1]}(F) =_{\text{def}} \Delta^{[k]}(F) \sqcup \overline{\Delta}(\Delta^{[k]}(F))$

3. $\Delta^{[*]}(F) =_{\text{def}} \operatorname{SUP}\left\{\Delta^{[k]}(F) \middle| k \in \mathbb{N}\right\}$

The following concepts of modality of a set Δ of finitary fuzzy deduction rules is sufficient that some of the following theorems can be proved.

Definition 9

1. *Δ is said to be* submodal *with respect to $\mathcal{L}$*

$$=_{\text{def}} \forall y \forall F \begin{pmatrix} y \in U \wedge F \in F\mathbb{P}(U) \\ \to \exists n \exists \delta \exists x_1 \cdots x_n \\ \begin{pmatrix} n \in \mathbb{N} \wedge \delta \in \Delta \wedge \delta \text{ is } n\text{-ary} \\ \wedge\, x_1, \ldots, x_n \in U \wedge \\ \overline{\Delta}(F)(y) = \\ = \min\begin{pmatrix} F(x_1), \ldots, F(x_n), \\ \delta(x_1, \ldots, x_n, y)\end{pmatrix}\end{pmatrix}\end{pmatrix}$$

2. *Δ is said to be* strongly submodal *with respect to $\mathcal{L}$*

$=_{\text{def}}$ *2.1. Δ is submodal with respect to U and*

2.2. $\forall y \forall F \begin{pmatrix} y \in U \wedge F \in F\mathbb{P}(U) \\ \to \exists k \begin{pmatrix} k \in \mathbb{N} \wedge \\ \Delta^{[*]}(F)(y) = \\ = \Delta^{[k]}(F)(y)\end{pmatrix}\end{pmatrix}$

Lemma 5

If Δ is strongly submodal with respect to $\mathcal{L}$, then

$$\Psi_\Delta(F) \sqsubseteq \Delta^{[*]}(F) \ .$$

Lemma 6

$$\Delta^{[*]}(F) \sqsubseteq \Psi_\Delta(F)$$

Theorem 7

If Δ is strongly submodal with respect to $\mathcal{L}$, then

$$\forall F \left(F \in F\mathbb{P}(U) \to \Psi_\Delta(F) = \Delta^{[*]}(F)\right) .$$

Proof

By lemma 5 and 6. ∎

Now, we are going to prove the compactness of Ψ_Δ. Therefore we start with the following lemma expressing the compactness of the mapping $\Delta^{[k]}$ where $k \in \mathbb{N}$.

Lemma 8

If Δ is submodal with respect to $\mathcal{L}$, then

$$\forall k \forall y \forall F \begin{pmatrix} k \in \mathbb{N} \wedge y \in U \wedge F \in F\mathbb{P}(U) \to \\ \exists F_{fin} \begin{pmatrix} F_{fin} \in F\mathbb{P}(U) \wedge F_{fin} \sqsubseteq F \\ \wedge\, F_{fin} \text{ is finite} \\ \wedge\, \Delta^{[k]}(F)(y) \leqq \Delta^{[k]}(F_{fin})(y)\end{pmatrix}\end{pmatrix}$$

Proof
By induction on k. ■

Theorem 9
If Δ is strongly submodal, then Ψ_Δ is compact, i. e.

$$\forall F \forall y \left(\begin{array}{l} F \in F\mathbb{P}(U) \wedge y \in U \rightarrow \exists F_{fin} \\ \left(\begin{array}{l} F_{fin} \in F\mathbb{P}(U) \\ \wedge F_{fin} \sqsubseteq F \wedge F_{fin} \text{ is finite} \\ \wedge \Psi_\Delta(F)(y) \leqq \Psi_\Delta\left(F_{fin}\right)(y) \end{array} \right) \end{array} \right)$$

Proof
By theorem 7 and lemma 8. ■

Consider a mapping

$$\Psi : F\mathbb{P}(U) \rightarrow F\mathbb{P}(U) .$$

We are going to investigate the problem under which conditions for Ψ there exists a set Δ of fuzzy deduction rules on U such that

$$\Psi = \Psi_\Delta .$$

We start with the following lemma

Lemma 10

$$\forall y \forall F \left(y \in U \wedge F \in F\mathbb{P}(U) \rightarrow \overline{\Delta}(F)(y) \leqq \Psi_\Delta(F)(y) \right)$$

Proof
By definition of Ψ_Δ and lemma 3. ■

Now, by using Ψ we construct a set Δ_Ψ of fuzzy deduction rules as follows. Therefore we fix an $n \in \mathbb{N}$. and an $\mathfrak{x} \in U^n$. Let $\mathrm{SET}_\mathfrak{x}$ be the set of elements of U belonging to $\mathfrak{x}$. For and arbitrary $y \in U$ and $F \in F\mathbb{P}(U)$ we define

$$F_\mathfrak{x}(y) =_{\text{def}} \left\{ \begin{array}{ll} F(y) & \text{if } y \in \mathrm{SET}_\mathfrak{x} \\ 0 & \text{if } y \notin \mathrm{SET}_\mathfrak{x} \end{array} \right\} .$$

Hence $F_\mathfrak{x}$ is finite and $F_\mathfrak{x} \sqsubseteq F$. Furthermore $\mathrm{supp}(F_\mathfrak{x}) = \mathrm{SET}_\mathfrak{x}$.

Definition 10

1. $\delta_\Psi^{n,F}(\mathfrak{x}, y) =_{\text{def}} \Psi(F_\mathfrak{x})(y)$ *where* $\mathfrak{x} \in U^n$ *and* $y \in U$
2. $\Delta_\Psi =_{\text{def}} \left\{ \delta_\Psi^{n,F} \middle| n \in \mathbb{N} \wedge F \in F\mathbb{P}(U) \right\}$

Lemma 11
If Ψ is compact then

$$\forall F \left(F \in F\mathbb{P}(U) \rightarrow \Psi(F) \sqsubseteq \Psi_{\Delta_\Psi}(F) \right) .$$

Proof
By definitions, lemma 3, and lemma 10. ■

For proving the following lemma 12 we need the concept of strong submodality of an operator Ψ.

Definition 11
Ψ is said to be strongly submodal *on U*
$=_{\text{def}}$
For every $y \in U, F \in F\mathbb{P}(U)$ there exist $n^0 \in \mathbb{N}$, $x_1^0, \ldots, x_{n^0}^0 \in U$ such that

$$\mathrm{Sup} \left\{ \min \left(\begin{array}{r} \Psi(F)(x_1), \ldots, \Psi(F)(x_n), \\ \Psi\left(H_{[x_1, \ldots, x_n]}\right)(y) \end{array} \right) \middle| \; n \in \mathbb{N} \wedge H \in F\mathbb{P}(U) \wedge x_1, \ldots, x_n \in U \right\}$$
$$= \min \left(\begin{array}{r} \Psi(F)(x_1^0), \ldots, \Psi(F)(x_{n^0}^0), \\ \Psi\left(\Psi(F)_{[x_1^0, \ldots, x_{n^0}^0]}\right)(y) \end{array} \right)$$

Lemma 12
If Ψ is a closure operator on $\mathcal{L}$ and Ψ is strongly submodal, then

$$\forall F \left(F \in F\mathbb{P}(U) \rightarrow \Psi_{\Delta_\Psi} \sqsubseteq \Psi(F) \right) .$$

Theorem 13
If Ψ is a strongly submodal compact closure operator, then

$$\forall F \left(F \in F\mathbb{P}(U) \rightarrow \Psi(F) = \Psi_{\Delta_\Psi}(F) \right)$$

Proof
By lemma 11 and lemma 12. ■

In the papers [5, 9, 10] the concept of closure operator includes the condition

$$(*) \quad \forall F \forall G \left(\begin{array}{l} F, G \in \mathfrak{F}(U) \rightarrow \\ \Psi(F \sqcup G) \sqsubseteq \Psi(F) \sqcup \Psi(G) \end{array} \right) .$$

This condition characterizes so-called topological closure operators (see, for instance [7, 10, 12, 17, 18]). If one has an "algebraic" closure operator Ψ (i. e. if Ψ is compact), then the condition above is too strong. More exactly speaking, the following theorem holds.

Theorem 14
*If Ψ is a strongly submodal compact closure operator, then Ψ satisfies the condition (*) if and only if there exists a system Δ of **1-ary** fuzzy deduction rules such that $\Psi = \Psi_\Delta$.*

4 Concluding Remarks

Because of restricted space in chapter 3 we could not develop a fuzzification of the whole theorem of G. BIRKHOFF, P. HALL, and J. SCHMIDT. In a forthcoming paper we shall continue the investigations started in the paper presented.

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