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THE RIGHT ANGLE: PRECISE NUMERICAL ORTHOGONALITY IN EIGENSTATES

By Julian V. Noble

SOLUTIONS OF THE SCHRÖDINGER EQUATION THAT PERTAIN TO DIFFERENT ENERGIES ARE ORTHOGONAL BY VIRTUE OF QUANTUM DYNAMICS. HOWEVER, WHEN WE OBTAIN SUCH

solutions numerically using library differential equation solvers, and when the inner product is defined by numerical quadrature, the result is not sufficiently orthogonal for certain purposes. Here, I show how to construct stable finite-difference schemes that preserve accurate numerical orthogonality of the solutions.

The problem

This column's prescription is about a formidable problem with a solution that turned out to be simple—once I got on the right track. It languished in my desk drawer for years, but now that I am coediting Computing Prescriptions, the idea seemed to cry out for resurrection.

We physicists investigate complex systems such as condensed matter, isolated atoms, atomic nuclei, or even isolated nucleons by giving them a whack and observing what comes out. A typical reaction of this sort is the photoelectric effect, where a photon incident on a metal causes an electron to be ejected (see Figure 1).

Such reactions conserve energy,

$$\epsilon_{\vec{k}} = \epsilon_0 + \omega, \quad (1)$$

where $\epsilon_{\vec{k}}$ is an ejected particle of momentum \vec{k} , ϵ_0 (usually negative) is the bound particle's energy, and ω is the probe-supplied energy.

Paradoxically, the rules of quantum mechanics let us hit a target hard enough to disrupt it (that is, to transfer lots of energy and momentum to it), while minimally perturbing it. The probability amplitude (or *matrix element*) describing such a process has the form

$$M_{fi} = \langle \Phi_f^{(-)} | V | \Phi_i^{(+)} \rangle, \quad (2)$$

where V is the (small and ephemeral) interaction of the probe with the target's constituents, and $\Phi_i^{(+)}$ and $\Phi_f^{(-)}$ are the probe and target's initial- and final-state wave functions. The probe and target are assumed noninteracting before and after V has acted.¹

To make headway, we often represent the probe particle's initial and final states as plane waves and pretend that a complicated system's constituents are effectively independent of each other. (Of course, the reaction could consume the probe particle, as in the photoelectric effect or meson absorption, in which case the probe's final state is the vacuum, or empty, state.) Thus, for example, we might think of a metal as a box of electrons, a nucleus as a sack of nucleons, or a nucleon as a bag of quarks. With such assumptions we reduce the computation to evaluating overlap integrals of the form

$$S_{fi} = \int d^3r \phi_k^{(-)*}(\vec{r}) \Gamma(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} \psi_0(\vec{r}), \quad (3)$$

where the local operator $\Gamma(\vec{r})$ might involve internal coordinates (such as spin or isospin), as well as the struck constituent's position. This integral's absolute square describes the probability of a transition, from the bound single-particle state ψ_0 to the unbound state $\phi_k^{(-)}$, induced by the particular probe in question. Sadly, representing the final state $\phi_k^{(-)}$ by a plane wave is not adequate in many problems. Rather, maintaining the dynamical orthogonality between ψ_0 and $\phi_k^{(-)}$ can be crucial.

One interesting experimental arrangement involves depositing a great deal of energy ω to the target system, without transferring much momentum \vec{q} . (In this context, "small \vec{q} " means "small relative to the inverse of the system's typical length scale"; "large ω " means "large relative to the binding of the initial particle, $-\epsilon_0$ ".) In such cases, the final state will be highly excited, which means its wave function will wiggle rapidly. The initial state, on the other hand, is typically smooth—ditto the operator $\Gamma(\vec{r})e^{-i\vec{q}\cdot\vec{r}}$. Figure 2 shows an example wave function corresponding to such reactions.

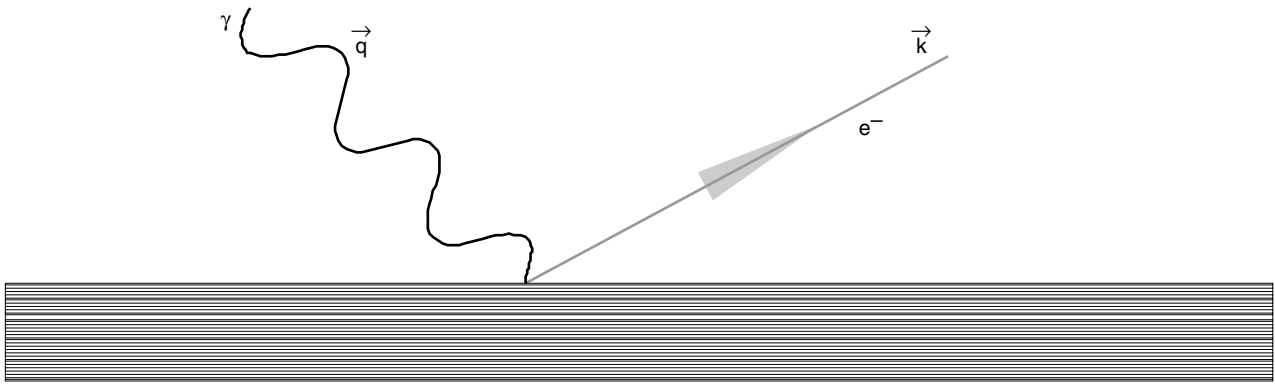


Figure 1. The photoelectric effect.

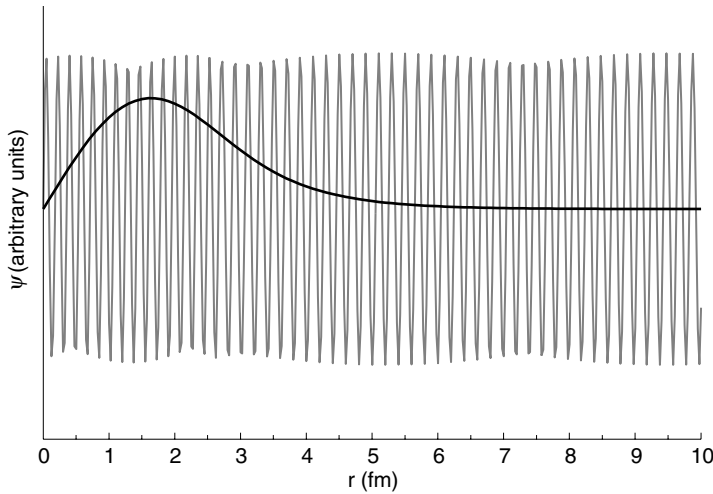


Figure 2. Bound and unbound solutions of the radial Schrödinger equation.

Of course, a highly oscillatory function's overlap with a smooth function must be small (the Riemann-Lebesgue Theorem guarantees this²); to get accurate results, the wave-function overlap must be zero—or at least exceedingly small on an appropriate scale.

Some years ago, my colleagues and I were trying to understand why the measured response in a case we (thought we) understood completely looked nothing like our calculations. (It's always a bad sign when this happens: it means you've overlooked something vital.) We eventually traced the problem to overlap integrals that should have vanished but came out relatively large in our calculation. Of course, because one function was smooth and the other oscillated rapidly, the Riemann-Lebesgue theorem guaranteed the integrals would be small in a sense. However, they were comparable in magnitude with the initial state's overlap with *any* oscillatory function that had the same asymptotic wavelength, $\lambda = 2\pi/k$ —that is, much too big. This defect led to manifestly spurious results at small $|\vec{q}|$. To achieve even modest extinction of such integrals (relative to that guaranteed by the Riemann-Lebesgue theorem), we had to use multiple precision and an exceedingly fine mesh.³

This brute-force approach got the job done, but it offended me: nuclear physics occurs on a scale of 10^{-13} cm (or in extreme cases, 10^{-14} cm), but we found we needed a mesh spacing 10^2 to 10^3 times smaller yet. Because no physical meaning can be attached to distances so much smaller than the system's typical length scale, we had obviously overlooked something fundamental. (Absorbing a low-energy π meson can eject a nucleon of about 130 MeV kinetic energy; the spacing between nodes in the ejected particle's wave function is something like 10^{-13} cm.) I eventually discovered an algebraic method to re-express the integrals so they vanished explicitly as $|\vec{q}| \rightarrow 0$.⁴ However, this technique proved inconvenient for numerical work; the rearranged integrals were much harder to compute (even if we did not have to evaluate them so precisely) because they contained derivatives of the

wave functions. I won't bore you with all the different things I tried, such as recasting the differential equations as integral equations (it works, but to get good precision you need big matrices—a major roadblock 24 years ago). However, it is worth remembering that inverting an $N \times N$ matrix takes $\mathcal{O}(N^3)$ time, whereas solving a differential equation on N mesh points takes $\mathcal{O}(N)$ time. The same objection applies to treating the differential equations as an eigenvalue problem with a sparse (tridiagonal) matrix—we can solve the equations in $\mathcal{O}(N)$ time, but getting the eigenvalues means solving an N th degree polynomial, where $N \sim 200$.⁵ (Note that J.H. Wilkinson and subsequent authors generally advocate the QR decomposition for the problem of computing all the eigenvalues and eigenvectors of real-symmetric tridiagonal systems.) Eventually I stumbled on the right approach by doing what I should have done from the start: analyzing why packaged differential equation solvers produced nonorthogonal wave functions (with reasonable mesh spacing).

Why “exact” states are orthogonal

To understand why functions that should be orthogonal

in theory aren't in practice, we must first understand why they are orthogonal in theory in the first place. For simplicity, I will illustrate with a bound and a continuum wave function in one spatial dimension, but the idea is general. If both initial and final states are solutions of the same Schrödinger equation

$$\begin{aligned} 0 &= \frac{d^2\phi}{dx^2} + (k^2 - U(x))\phi \\ 0 &= \frac{d^2\psi}{dx^2} + (-\kappa^2 - U(x))\psi, \end{aligned} \quad (4)$$

we can multiply the complex conjugate of the first equation by ψ , multiply the second equation by ϕ^\dagger , and then subtract

$$\frac{d^2\phi^\dagger}{dx^2}\psi - \phi^\dagger\frac{d^2\psi}{dx^2} \equiv \frac{d}{dx}\left(\frac{d\phi^\dagger}{dx}\psi - \phi^\dagger\frac{d\psi}{dx}\right) = -(k^2 + \kappa^2)\phi^\dagger\psi. \quad (5)$$

Remember that the potential $U(x)$ must be a real function—it doesn't work if $U^\dagger \neq U$. Now, integrate from the first boundary a to the second, b , to get

$$\int_a^b dx \frac{d}{dx}\left(\frac{d\phi^\dagger}{dx}\psi - \phi^\dagger\frac{d\psi}{dx}\right) = -(k^2 + \kappa^2)\int_a^b dx \phi^\dagger(x)\psi(x). \quad (6)$$

The left side is the integral of the perfect derivative of the "Wronskian" of ϕ^\dagger and ψ , so we integrate it and evaluate the result at the ends. With appropriate boundary conditions (imposing appropriate boundary conditions at the endpoints is crucial to the differential operator's hermiticity), the Wronskian vanishes at both ends, hence

$$\left(\frac{d\phi^\dagger}{dx}\psi - \phi^\dagger\frac{d\psi}{dx}\right)\Big|_a^b = 0 = -(k^2 + \kappa^2)\int_a^b dx \phi^\dagger(x)\psi(x). \quad (7)$$

Because $k^2 + \kappa^2 \neq 0$, the inner product must vanish:

$$\langle\phi|\psi\rangle \stackrel{df}{=} \int_a^b dx \phi^\dagger(x)\psi(x) = 0. \quad (8)$$

That is, a Hermitian differential operator's eigenstates, belonging to distinct eigenvalues, are orthogonal. The definition (Equation 8) of inner product for the Hilbert space of wave functions emerges automatically.

Why the computed states were not orthogonal

Because the operator

$$H = U(x) - \frac{d^2}{dx^2} \quad (9)$$

is Hermitian (with the correct boundary conditions), it is guaranteed that the finer we take the integration mesh and the closer we come to the exact solutions of the Schrödinger equation, the smaller the numerical approximation to the overlap integral will be:

$$\int_a^b dx \phi^\dagger(x)\psi(x) \approx \sum_{n=-N}^N w_n \phi^\dagger(nh)\psi(nh), \quad (10)$$

where w_n are the weights and $x_n = nh$ the points of a uniformly-spaced quadrature formula such as the trapezoidal rule or Simpson's rule. (There is no point in using either adaptive quadrature or a higher-order quadrature rule because the functions to be integrated were computed on a uniform mesh by standard ODE solvers. No interpolation scheme can provide more information than is present already; the only way to achieve better precision is a finer mesh.) The surprise was just how fine the mesh had to be before we got acceptably small overlaps.

To discover what went wrong, recapitulate every step in computing the wave functions and their overlap integral. Begin with the wave functions. To be specific, let $0 \leq x < \infty$, and take the potential $U(x)$ to be well behaved (in the sense of being finite for finite x and vanishing sufficiently fast as $x \rightarrow \infty$). We will also take the potential to be deep enough to support at least one bound state. Figure 3 shows a typical 1-dimensional potential.

Consider first the bound state ψ . It is proportional to x^1 as $x \rightarrow 0$, so we get the numerical solution by

- Picking a value for κ
- Starting at $x = 0$
- Setting $\psi(0) = 0$ and $\psi'(0) = 1$
- Integrating rightward until $U(x)$ is negligible
- Determining the coefficient $A(\kappa)$ of the exponentially increasing term; the solution has the form $\psi(x) \rightarrow Ae^{\kappa x} + Be^{-\kappa x}$
- Employing an appropriate root-finding strategy to find the zero of $A(\kappa)$

This method is essentially that of Max Krell and Torleif Ericson.⁶ Most canned eigenvalue solvers use the "shooting"

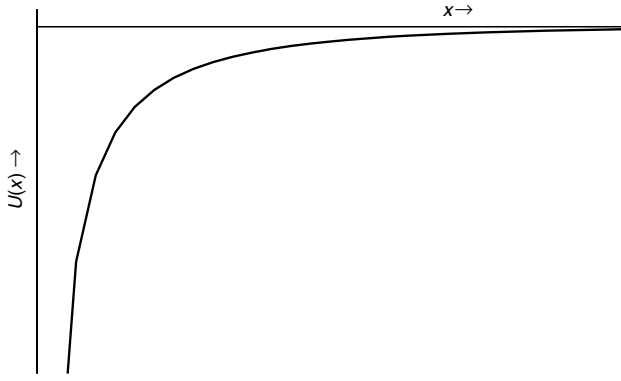


Figure 3. Attractive Hulthén potential on interval $(0, \infty)$.

method,^{7,8} which integrates from $x = 0$ rightward, then from some large X leftward (assuming the solution there is $e^{-\kappa x}$), and matches the two at a convenient intermediate point.

Integrating the continuum wave function is substantially simpler. With a spherically symmetric potential, we start at the center and integrate outward until the solution oscillates with constant amplitude.

Finally, we compute the overlap integral, where we get a result too large by several orders of magnitude, unless we use an extremely fine mesh and are exceedingly careful of round-off error in solving the ODEs.

Hermiticity and orthogonality in difference equations

When functions that should be exactly orthogonal (because they are eigenvectors of a Hermitian operator) are not, we must suspect that perhaps the operator was not Hermitian after all. This, in fact, turned out to be the case with numerical solutions of the Schrödinger equation, and for a surprising reason. Consider how we integrate numerically the differential equation

$$\psi'' + f(x)\psi = 0, \quad (11)$$

where

$$f(x) = -U(x) - \kappa^2. \quad (12)$$

First we discretize the interval, $x \rightarrow xn = hn$, so that h is the step size δx . Next, we expand in Taylor's series

$$\begin{aligned} \psi_{n+1} &= \psi(x_n + h) = \psi_n + h\psi'_n + \frac{1}{2}h^2\psi''_n + \dots \\ \psi_{n-1} &= \psi(x_n - h) = \psi_n - h\psi'_n + \frac{1}{2}h^2\psi''_n - \dots \end{aligned} \quad (13)$$

and add, to get

$$\Delta^2 \psi_{n-1} \stackrel{df}{=} \psi_{n+1} + \psi_{n-1} - 2\psi_n = h^2\psi''_n + \frac{1}{12}h^4\psi_n^{(iv)} + O(h^6). \quad (14)$$

The simplest algorithm stops at the second-order term

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n \approx h^2\psi''_n, \quad (15)$$

which, by virtue of the differential Equation 11, we can rewrite:

$$\Delta^2 \bar{\psi}_{n-1} = -h^2 f_n \bar{\psi}_n. \quad (16)$$

(I use the notation $\bar{\psi}_n$ to distinguish the values we compute at the mesh points from the “true” values $\psi(nh)$.)

The technique we used to get the orthogonality of the differential equation's solutions also works for the difference equation. The identity

$$\bar{\phi}_n^\dagger \Delta^2 \bar{\psi}_{n-1} - \bar{\psi}_n \Delta^2 \bar{\phi}_{n-1}^\dagger \equiv \Delta(\bar{\phi}_n^\dagger \Delta \bar{\psi}_{n-1} - \bar{\psi}_n \Delta \bar{\phi}_{n-1}^\dagger) \quad (17)$$

lets us sum (rather than integrate) the discrete equivalent of the Wronskian Equation 6 to get an orthogonality relation:

$$\langle \phi | \psi \rangle_{disc} \stackrel{df}{=} h \sum_n \bar{\phi}_n^\dagger \bar{\psi}_n = 0, \quad (18)$$

which is now a sum rather than an integral. The orthogonality now depends on the discrete Wronskian

$$W_n = \bar{\phi}_n^\dagger \Delta \psi_{n-1} - \psi_n \Delta \bar{\phi}_{n-1}^\dagger \quad (19)$$

vanishing at the ends of the summation interval.

Let's now study the solutions of the difference Equation 16. For values of n such that $U(nh)$ is negligible, the asymptotic bound-state equation becomes $\Delta^2 \bar{\psi}_n = h^2 \kappa^2 \bar{\psi}_n$, whose solutions are of the form $\bar{\psi}_n = \lambda^{\pm n}$, rather than $e^{\pm \kappa n h} \equiv (e^{\pm \kappa h})^n$. Solving the resulting quadratic equation

$$\lambda - 2 + \lambda^{-1} = h^2 \kappa^2, \quad (20)$$

we find two real roots,

$$\lambda = 1 + \frac{h^2 \kappa^2}{2} \pm h \kappa \sqrt{1 + \frac{h^2 \kappa^2}{4}}, \quad (21)$$

one of which (with the + sign) is larger than 1; the other (with the - sign) is the inverse of the first, hence smaller than 1. We always refer to the root with the + sign as λ .

A similar relation holds for the positive-energy eigenvector $\bar{\phi}_n$. The solution has the asymptotic form $\bar{\phi}_n = \lambda^{\pm n}$ where now

$$\lambda = 1 - \frac{h^2 k^2}{2} \pm i h k \sqrt{1 - \frac{h^2 k^2}{4}}. \quad (22)$$

Referring to the root with the + sign as λ , note that $\lambda^* \lambda = 1$ —that is, $\lambda = e^{i\theta}$ where θ is some real phase angle. In other words, the positive-energy solutions oscillate sinusoidally, as we expect.

Here is what went wrong when we computed overlap integrals using Simpson's rule to numerically integrate the solutions provided by packaged differential equation solvers:

1. The correct (discretized) inner product, for which we might expect orthogonality, is the trapezoidal rule, not Simpson's rule.
2. The negative-energy eigenvalue the library routine found is not appropriate to the (discretized) Hermitian eigenvalue problem.

The second item is the key: packaged routines solve for eigenvalues by asymptotically matching to the exponential $e^{-\kappa x}$ and by varying the mesh spacing. The accurate eigenvalues they obtain in this way are wrong for our purpose. As $h\kappa \rightarrow 0$, the discretized system's solution approaches that of the continuous system. But even for a relatively fine mesh—for example, $h\kappa \approx 0.025$ —the “true” eigenvalue differs sufficiently from that appropriate for the difference equation to spoil the vanishing of the approximate overlap integral computed by numerical quadrature.

An example

Having exactly soluble examples that we can compare with numerical computations is always good. A well-known case is the Hulthén potential⁹

$$U(x) = \frac{U_0}{e^{ax} - 1}, \quad (23)$$

for which the Schrödinger equation becomes

$$\psi'' + \left(k^2 + \frac{U_0}{e^{ax} - 1} \right) \psi = 0. \quad (24)$$

The substitutions $\xi = e^{-ax}$ and $\psi(x) = e^{-\kappa x} \chi(x)$ transform the equation to the hypergeometric equation (to save writing, let $a = 1$):

$$\xi(1-\xi) \frac{d^2 \chi}{d\xi^2} + (1-\xi)(2\kappa+1) \frac{d\chi}{d\xi} + U_0 \chi = 0. \quad (25)$$

The solution that goes to zero as $\xi \rightarrow 1$ (that is, $x \rightarrow 0$) is

$$\psi(x) = e^{-\kappa x} (1 - e^{-x}) F(\alpha, \beta, \gamma; e^{-x}), \quad (26)$$

where F is a hypergeometric function, and α , β , and γ can be read off from the differential equation.¹⁰ For this solution to remain finite as $\xi \rightarrow 0$ (for $x \rightarrow \infty$), the hypergeometric series must terminate. This is possible for integers $1 \leq n \leq \sqrt{-U_0}$, giving

$$\kappa_n = \frac{-U_0 - n^2}{2n} \quad n = 1, 2, \dots \quad (27)$$

We get the continuum states by substituting $\kappa \rightarrow -ikx$ (and, of course, the hypergeometric series is no longer required to terminate).

As a test case, I have chosen $\kappa = 0.25$ and $-U_0 = 2\kappa + 1 = 1.5$ (the potential thus has only one bound state). I used the second-order difference formula, Equation 16, stepping through to $x = 20$ with a uniform step size $h = 0.1$. The initial conditions were $\bar{\psi}_0 = 0$, $\bar{\psi}_1 = h$. At each step, I renormalized by multiplying $\bar{\psi}_n$ and $\bar{\psi}_{n-1}$ by λ^{-1} ; because the solution (for an arbitrary κ) will approach $A\lambda^n + B\lambda^{-n}$ for large positive n , this has the effect of isolating the coefficient $A(\kappa)$ at the end of the iteration. I then used a hybrid binary search regula falsi algorithm to locate the root, finding $\kappa = .24895\dots$, which differs considerably from the value (0.25) we expect in the continuum limit. Finally, I evaluated the bound state's discrete inner product with the continuum wave functions I got by using the same difference equation (see Table 1).

The problem that originally interested me involved realistic nuclear one-body potentials, whose shape was of the Woods-Saxon form,

$$V(r) = \frac{V_0}{e^{(r-R)/a} + 1}. \quad (28)$$

When this column was nearly finished, I discovered some

Table 1. Overlap integrals of 1s solutions in the Hulthén potential.

k^2	$\varphi = \sin(kr)$	$\bar{\varphi}_n^{(-)}$
0.5	1.33041760403370E1	-1.55679719537791E-3
1.0	1.83902528627110E1	-2.34783137074259E-3
1.5	1.95130603513789E1	2.59341193983747E-3
2.0	1.92221486833249E1	1.03566115867689E-2

Table 2. Overlap integrals in the Woods-Saxon potential.

$l = 1$ (1 p)		$l = 0$ (2 s)		
$k(\text{fm}^{-1})$	$\varphi = kr j_l(kr)$	$\bar{\varphi}_{n,1}^{(-)}$	$\varphi = kr j_0(kr)$	$\bar{\varphi}_{n,0}^{(-)}$
1.0	1.5877E0	8.88852E-6	1.5877E0	-2.9541E-7
1.5	4.8954E-1	7.9293E-6	4.896E-1	8.7053e-8
2.0	-7.4794E-2	7.3306E-6	-7.4787E-2	8.2968e-9

Parameters: $R = 5 \text{ fm}$, $a = 0.5 \text{ fm}$, $V_0 = -50 \text{ MeV}$, $h = 0.1 \text{ fm}$

binders dense with 11 × 15 inch fanfold output dealing with this very problem, dating from the late 1970s. Discovering that my memory was in pretty good shape was nice; the program I just wrote to do the Hulthén case is virtually identical to my old program, except for the change of programming language. Just for fun, I include some of my old results, obtained using the same second-order integration algorithm, Equation 16 (see Table 2). They pertain to 1p and 2s bound state wave functions.

The observant reader might inquire why the extinction factor in my recent calculation is only three or four orders of magnitude, whereas that found in my 24-year-old calculation was six or seven orders. The answer is that the IEEE 8-Byte double-length real representation supplied by the math coprocessor in Intel and AMD chips is equivalent in precision to the single-length real (60 bits) on the CDC 6600 mainframe that I used in the 1978 calculation. And I used double precision (120 bits on the CDC). To get the equivalent on a PC I would have to define a quadruple length (that is, 16-Byte) floating-point number representation, and redefine all the arithmetic routines (in machine language). This looks too much like work, so I'll put it off to another day. Maybe the sixth Tuesday of next month.

There is a moral here: When we compute such amplitudes we are actually differentiating numerically a function that would be hard to differentiate symbolically. To get reasonable results we therefore need unusually high precision. The good news is that today's prescription lets us use a coarse

mesh, speeding up the calculation 50- to 100-fold, even using quadruple precision floating-point arithmetic.

Higher-order schemes

I hope I haven't left you with the (erroneous) impression that precise orthogonality requires the trapezoidal rule for our overlap integral, combined with the second-difference scheme, Equation 16. Here are two ways to use higher-order integration formulas, which automatically yield appropriate discrete inner products.

First, the popular Numerov algorithm approximates the fourth-order term of Equation 14 with the second difference of $-f_n \bar{\psi}_n$.^{11,12} This yields

$$\Delta^2 \psi_{n-1} \approx -h^2 f_n \psi_n - \frac{1}{12} h^2 \Delta^2 (f_{n-1} \psi_{n-1}), \quad (29)$$

which we can rewrite as

$$\Delta^2 \left[\left(1 + \frac{1}{12} h^2 f_{n-1} \right) \psi_{n-1} \right] \approx -h^2 f_n \psi_n. \quad (30)$$

The point of this algorithm is that it is fourth order but requires no more work than the second-order algorithm. Interestingly, this algorithm yields the same (trapezoidal rule) inner product as the second-order formula, although the eigenvalue condition is slightly modified, because now

$$\lambda = 1 + \frac{h^2 \kappa^2}{2 \left(1 - \frac{h^2 \kappa^2}{12} \right)} + h \kappa \sqrt{1 + \frac{h^2 \kappa^2}{6}} \left(1 - \frac{h^2 \kappa^2}{12} \right) \quad (31)$$

and similarly for the continuum version.

An alternate approach, which I leave as a homework exercise (professors just love to say this), is to derive the difference equation from a variational principle. Defining an action

$$\mathcal{A} = \sum w_n \left(\frac{\hbar^2}{2m} (D\bar{\psi}_n^\dagger)(D\psi_n) + \bar{\psi}_n^\dagger (V_n - E)\bar{\psi}_n \right), \quad (32)$$

where $D\psi_n$ represents an appropriate difference scheme for approximating the derivative ψ' , we can vary with respect to the $\bar{\psi}_n^\dagger$ to get a difference version of the Schrödinger equation, of any order we like. The action's symmetry ensures that the solutions will be orthogonal—if we apply Hermitian boundary conditions to get the eigenvalues.

Let me summarize: this prescription describes a numerical difficulty that initially seemed baffling. Careful analysis revealed that we were not calculating what we thought—namely, the inner product of two eigenstates of a Hermitian operator corresponding to different eigenvalues—but something else entirely. To fix this, I redefined the problem to be properly Hermitian on a discrete space. The key lay in computing eigenvalues appropriate to the asymptotic behavior of solutions of a difference equation.

At first I considered this revision a bit of a cheat. But then I realized that the potentials we use to model our target systems are not known in the kind of detail that would justify the effort of an “exact” solution. That is, in practical terms the (much easier) substitute problem cannot be distinguished from the original, so why not use it? Physicists, after all, believe devoutly in the conservation of energy, especially their own. ☞

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