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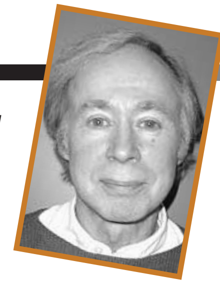

On Hearing the "Shape" of a Vibrating String

By Nicholas Giordano

Nearly four decades ago, Mark Kac first posed what is now a famous question in mathematical physics.[1] Stated simply, this question is, "Can you hear the shape of a drum?" In a delightful paper, Kac discussed the problem's meaning and importance, along with its connections to a surprisingly large number of interesting questions and people in mathematical physics. This paper is an example of mathematical physics at its best. While he credited the original articulation of the problem to Salomon Bochner and Lipa Bers, Kac seems to be the first to recognize the many connections to other areas. Physicists and mathematicians whose work has touched on this problem include Hendrick Lorentz, David Hilbert, and Hermann Weyl.

Here is another way to state this problem. A simple drum consists of a vibrating membrane whose perimeter is held rigidly in some particular shape. You can then ask if knowledge of all the normal mode frequencies of such a membrane, which you could obtain by "listening to the drum," uniquely determines the shape. About a decade ago, mathematicians Carolyn Gordon, David Webb, and Scott Wolpert answered this question by constructing pairs of drums that have different shapes but the same normal mode spectra.[2] Thus, they proved that you cannot hear a drum's shape.

In this article, I discuss a related question concerning vibrating strings. Many musical instruments, including the guitar, piano, and violin, employ vibrating strings. A string's vibrational modes are well known; they are just the usual standing waves. For an ideal string that is fixed at both ends, the normal mode frequencies follow the familiar pattern of a fundamental f_1 and a series of harmonics $f_n = nf_1$ where n is an integer. All real strings possess essentially this set of normal modes, but we know that the instruments I just mentioned do not sound at all alike. So, a musical tone is clearly more than merely the pattern of normal modes. A crucial ingredient is how the player sets the string into motion. A guitar string is plucked, a piano string is struck by a felt-covered hammer, and a violin string is excited by the stick-slip frictional interaction with a bow. These excitations give rise to very different sounds, even though the underlying pattern of normal modes is the same.

So, from the musical tone that a vibrating string produces, can you discern the manner in which the string is excited? Or, to paraphrase Kac, can you hear the "shape" of a vibrating string?

As you'll see, combining Fourier analysis with symmetry considerations can give us insight as to how the spectral composition of a tone (what you hear) reflects the nature of the excitation (the string's shape). This problem nicely illustrates the power of Fourier analysis with a number of examples that fit easily into the classroom, particularly as part of a course on physics simulation methods.

After a few preliminaries on computing a string's vibrational motion, I consider the case of the guitar. Here the initial excitation is a simple plucked waveform, and I explore how a tone's spectrum depends on where the player plucks a string. I then consider two other instruments, the piano and the clavichord. The latter, a relative of the piano,[3] will probably not be familiar to most readers; it is of interest here because its strings are excited in a rather unusual manner.

Vibrating strings: Solving the equation of motion

Figure 1 shows a sketch of an acoustic guitar. One end of each string passes over a fret on its way to a tuning peg, while the other end passes over the bridge, which is attached to the soundboard. This board is a thin piece of wood that moves in response to forces that the strings exert. The soundboard is a sort of speaker and is responsible for nearly all the sound that is produced. Here we focus on the string, so we treat the string's motion in detail and treat the board in a simple fashion, which allows for a direct connection between the string's motion and the musical tone.

For the string, we start with the usual wave equation, the two left-most terms in Equation 1:

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$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - \frac{\pi E r^4}{4\mu}\frac{\partial^4 y}{\partial x^4} - \alpha\frac{\partial y}{\partial t}. \quad (1)$$

We add additional terms to account for string stiffness (proportional to the Young's modulus E) and internal damping (proportional to α),[3] where T, μ, and r are the tension, mass per unit length, and string radius. For musical-instrument strings, the stiffness and damping terms are both small, but they are essential if we want our calculation to produce realistic musical tones. Equation 1 does not include any external forces and thus applies only to a freely moving string.

Solving this equation numerically using an explicit finite-difference method is convenient and straightforward. This approach, which is described in detail elsewhere,[4–7] works as follows. First, we write the derivatives of the string displacement $y(x, t)$ in Equation 1 in finite-difference form in terms of $y(i, m) \equiv y(i\Delta x, m\Delta t) \equiv y(x, t)$, where we divide space and time into discrete elements Δx and Δt, with i and m integers. This converts Equation 1 to a system of algebraic equations, which we can rearrange to express the string displacement at the "next" time step $y(i, m + 1)$ in terms of the string displacements at previous time steps. Given proper initial conditions, this procedure can be iterated forward in time (for all i) to obtain the string's time-dependent motion. With proper attention to numerical stability,[6] this approach provides an accurate, convenient way to deal with the wave equation. For a typical (and realistic) string with a length of approximately 0.5 meter and $c \sim 200$ meters per second, the number of discrete finite-difference elements can be approximately 50, with a time step of $\sim 0.5\ \delta x/c$, which turns out to be approximately 3×10^{-5} seconds. Such simulations run in nearly real time on a typical personal computer.

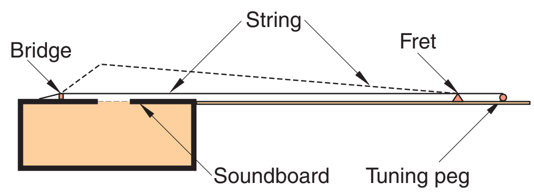

Figure 1. A rough schematic of an acoustic guitar (not to scale). For clarity, the schematic shows only one string, but a typical guitar contains six. The dashed lines show a plucked profile just before the string is released.

The boundary conditions at the string's end warrant some discussion. Typical textbook treatments of standing waves on strings usually assume that the ends are held rigidly; that is, $y = 0$ at the ends. However, from a purely numerical viewpoint, this boundary condition alone is not enough to deal with the fourth-order derivatives in Equation 1. It is tempting to assume the string is clamped at the ends so that $y = 0$ at the ends and beyond (that is, on the bridge). While this approach can work, it is more convenient to employ hinged boundary conditions at the string's ends.[4,5] At the end held at the fret (on the right in Figure 1), this means that $y(x_f) = 0$ at the fret (x_f) together with $y(x_f + dx) = -y(x_f - dx)$. The situation at the bridge is similar, except that because the bridge can move (as the soundboard moves), the string pivots about y_{bridge} rather than $y = 0$ (as we consider in more detail later).

As I just mentioned, the soundboard must move; otherwise, we would have no sound to listen to. We will use a very simple model of soundboard motion and sound production. We assume that the soundboard moves according to

$$v_b = \frac{dy_b}{dt} = \frac{F_s}{Z}, \quad (2)$$

where v_b and y_b are the soundboard's velocity and position, F_s is the force the string exerts on the board, and Z is the soundboard's mechanical impedance. F_s is due to the tension's perpendicular component and is just T times the slope of the string where it meets the bridge. Z is usually discussed in the frequency domain, but for simplicity we will take Z to be a constant; this is also a good approximation for real soundboards.[8] In the end, we will want to consider the sound the vibrating board produces; to a reasonably good approximation, this sound is simply proportional to v_b.[9]

Plucked guitar strings

To begin a simulation, we need to choose the proper initial string profile $y_i(x)$. For a guitar, this is a simple plucked waveform. That is, at time $t = 0$ the string has the profile shown as the dashed curve in Figure 1. This is two straight lines that begin at zero displacement ($y = 0$) at both the bridge and the fret, and meet where the player plucks the string. For snapshots of the resulting string profiles as a function of time after the pluck, see my article "The Physics of Vibrating Strings."[6] When the player releases the string, the kink that was originally at the player's finger splits into two kinks that propagate in opposite directions. When they reach the string's ends they

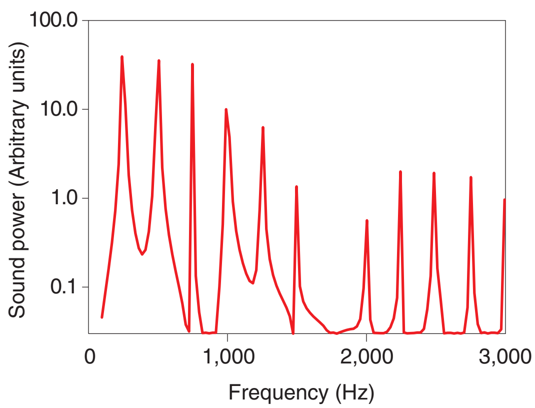

Figure 2. A guitar string's spectrum (for the note B, just below middle C, with a fundamental frequency near 240 Hz) under normal playing conditions. I calculated this by performing a fast Fourier transform on the initial 1,024 points of the calculated sound waveform (the time step for the calculation was 5×10^{-5} seconds). This is why the peaks have finite widths.

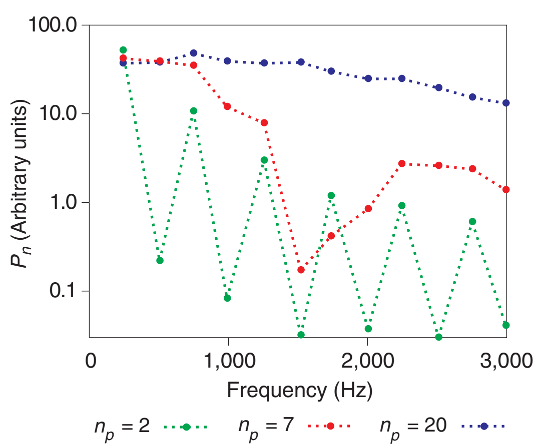

Figure 3. Magnitudes of the normal modes for a guitar string plucked at various distances L/n_p from the bridge. I obtained these results by taking the peak values from fast Fourier transform spectra such as the one in Figure 2.

are reflected. So, the string's time-dependent motion consists of two kinks moving back and forth, which leads to a time-dependent force on the soundboard.[3,6] From this force we can calculate the bridge's motion using Equation 2, thereby producing the sound's spectrum (see Figure 2). This power spectrum's peaks correspond to the string's fundamental mode of vibration and its harmonics. However, the interesting point is the variation of the peak power at each harmonic P_n as a function of harmonic number n. We see that P_n has a pronounced minimum for n near 7 and then increases at larger n.

For a simple musical tone, the fundamental's frequency determines the pitch. However, the tone color or timbre comes from the relative strengths of the harmonics, so the dip in P_n near $n = 7$ is important. To understand this dip's origin, consider the spectrum of the same string plucked in other places. Figure 3 shows results for the peaks in the power spectra P_n when our guitar string is plucked at three different spots. The curves in Figure 3 correspond to strings plucked a distance L/n_p from the bridge, where L is the string length and the string is plucked a fraction $1/n_p$ of its length from one end.

The behavior with $n_p = 2$, a string plucked at its center, is perhaps the most striking. The modes alternate in strength, with the fundamental and the odd harmonics being much stronger than the neighboring even harmonics. The initial excitation's symmetry helps us understand why this is so. The initial string profile $y_i(x)$ in this case is symmetric with respect to the string's center. So, a Fourier expansion of $y_i(x)$ will contain only spatially symmetric modes (that is, symmetric with respect to the string's center). These will be the harmonics with n odd. All modes with n even are described by a waveform that is antisymmetric with respect to the string's center. This suppresses their size (they would in fact be vanishingly small in a numerical calculation with $\Delta x \rightarrow 0$). Figure 3 shows precisely this behavior. We can easily hear this suppression of the even harmonics. For a real guitar, moving the plucking point only 1 cm from the string's center produces a very noticeable change in the tone color.

The result in Figure 3 for $n_p = 7$ corresponds to the spectrum in Figure 2. For this pluck at $L/n_p = L/7$, the Fourier components at $n = 7, 14, \ldots$ are suppressed. We can understand this intuitively. The initial pluck has a maximum at $L/7$, so modes with nodes at this point will not be excited. Mathematically, we can demonstrate this in several ways. One is to simply compute analytically the Fourier amplitudes either in the usual way (by integrating the product of $y_i(x)$ and the appropriate sin functions) or by using a symmetry argument similar to the one we used earlier for $n = 2$. (I'll leave that argument's construction to you. Hint: First consider the case $n = 3$ and then proceed to higher n by induction.)

The third case in Figure 3 is for a string plucked close to the end, $n_p = 20$. Here, we expect a dip at the 20th harmonic, which is beyond the frequency range shown in Figure 3. However, the important result is that P_n now decays very slowly with n, so that the harmonics are much stronger relative to the fundamental. This is what gives the distinctive tone color, a "twang," for a guitar that is played in this manner.

Our results for a plucked guitar string indicate that we can indeed hear the string's shape; the plucking point and the tone color are directly connected. However, there are other ways to put a string into motion, as we explore in the next two sections.

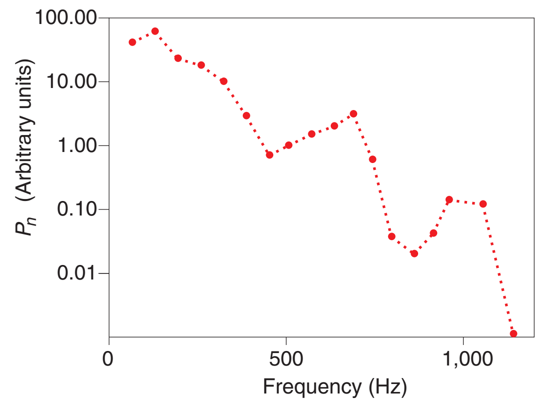

Figure 4. Power at the spectral peaks for a piano string for the note two octaves below middle C.

The piano

Very roughly speaking, a piano is similar to a guitar. One end of each string runs over a bridge that is attached to a soundboard. However, rather than being plucked, the strings are struck by felt-covered hammers. Modeling this hammer–string collision requires an equation of motion for the hammer (which is quite simple) and a description of the collision force (which is only a little more work). I don't have space to discuss these here; detailed discussions of them appear elsewhere.[4,6] One interesting aspect of this calculation is the need to model a flexible object, the hammer's felt surface.

Some typical results appear in Figure 4, which plots the peaks in the power spectrum P_n for the note commonly called C2. This note is two octaves below middle C and has a frequency near 65 Hz. The calculation's parameters came from my grand piano and recent experiments. (I performed this calculation for a steel string on a typical piano. The relevant parameters for such a string and for the other calculations in this article are at www.physics.purdue.edu/~ng.) As with the guitar, we find that P_n exhibits dips, which here are near $n = 7$ and 14. Knowledge of the guitar might lead us to guess that these dips are connected with the place where the hammer strikes the string; this is indeed correct. The hammer strike point is approximately $L/7$ from the string's end, so this case is closely analogous with that of the guitar.

However, closer comparison of the guitar spectrum with $n_p = 7$ and the piano results reveals some differences. In particular, for the guitar, P_n decreases more slowly with n, so the dip at $n = 7$ is narrower (as are the ones at higher n). Also, the (relative) power at frequencies beyond the first dip is larger for the guitar. These differences are easy to understand intuitively if we recall that a piano hammer is a "soft" object, so its collision with the string is "gentler" than is the case for a plucked guitar string.

Stating this in another, more or less equivalent, way is instructive. For the guitar, the initial string profile $y_i(x)$ completely specified the initial conditions. For the piano, the initial conditions are a little more complicated. Strictly speaking, the initial condition is an undisplaced string at rest just before the hammer impact. But this viewpoint is not very useful because it ignores what the hammer is about to do. Considering the string's state just after it loses contact with the hammer is more useful. At that instant, the string has some nonzero displacement $y_i(x)$ and a nonzero velocity $v_i(x)$. So, we must generalize this article's central question to include $v_i(x)$. That is, for the piano, the string's initial shape does not fully determine the sound—that is, the behavior of P_n. To hear a piano string's state, we must also "hear" the initial string velocity.

The clavichord

After considering the guitar and the piano, you might imagine that we have

Exercises and Projects

If you are interested in pursuing calculations of realistic musical instrument tones, here are two suggested problems.

Fretless stringed instruments

The article "Musical Acoustics and Computational Science" describes a fairly detailed and realistic-sounding calculation of tones for a guitar (or other similar instrument).[1] In this case, one end of the string is fastened to the bridge and the other passes over a fret (see Figure 1 in the main article). However, some stringed instruments do not have frets—for example, some bass guitars.

No one has ever computationally explored the tones that such an instrument produces; an interested reader could do so by building on the calculation in the article I just mentioned.[1] The major change would be to model the extra damping introduced when the player's finger holds down one end of the string. You could do this by numerically "attaching" the string's end to a mechanical impedance similar to the one I used to model the soundboard in Equation 2 in the main article. Or, you could introduce a strong damping force, proportional to the string's velocity, at the string's end—that is, acting only on the string's last discrete numerical element. With either approach, you will have to adjust the damping's strength to obtain a realistic tone.

Bending

The guitar-playing technique called *bending* involves shifting a note's frequency by moving the string sideways along the fret. This increases the frequency by increasing the string's tension. (The tremolo bar on guitars such as the Fender Stratocaster can produce a similar frequency shift.)

An interesting computation would be to model this by starting with a normal calculated guitar tone[1] and then varying the tension according to some prescribed manner. To investigate the size of this tension change and the way that it depends on time during the note, you would have to listen to the calculated tones. The industrious reader could go further by attempting to calculate the magnitude of the tension change produced by pushing the string's end a few millimeters or more along a fret, using a steel guitar string's elastic properties.[2] The goal would be to produce realistic sounding tones with physically accurate modeling. The truly industrious reader would compare his or her calculated tones in detail (through either listening tests or the Fourier analysis technique I employed in the main article) with those from a real instrument.

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exhausted the number of truly different ways to excite a string. However, this is not the case. Figure 5 shows results for the clavichord. This somewhat obscure instrument is a relative of the harpsichord and an ancestor of the piano[3,10] (and was, according to some accounts, Johann Sebastian Bach's favorite instrument). Its spectrum is distinctly different from that of the guitar or piano. For the clavichord, P_n falls more rapidly for small n, and there are no dips corresponding to a plucking point or a hammer strike point. From our understanding of the guitar and the piano, we might have imagined that such dips must always occur. The clavichord results show that this is not true.

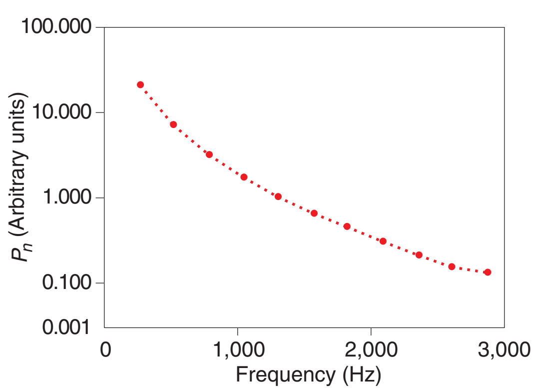

Figure 5. The calculated power at the spectral peaks for a clavichord string for middle C, which has a fundamental frequency near 262 Hz.

To see how the clavichord avoids such dips, we must consider how its strings are set into motion. Like the piano, the clavichord is a keyboard instrument containing many strings, all of which pass over a bridge attached to the soundboard. Near each key's end is a *tangent*, a narrow metal strip (or blade) that is perpendicular to the corresponding string. When the player depresses the key, the tangent contacts the string and displaces it a small amount (a few millimeters). The tangent's force holds the string's end in this new position, and the tangent stays in contact with the string until the player releases the key. This contact causes the portion of the string between the tangent and the bridge to vibrate (the string's other section is damped so that it does not vibrate appreciably). So, what sets the clavichord string into motion is the abrupt motion of one of its ends. Because the string's end is normally a node of the vibrational mode, this is not an efficient way to excite a vibration. For this reason, the clavichord produces very little sound; it is a rather "quiet" instrument. However, the player directly controls each note's volume, something that was not possible in other keyboard instruments until the piano was invented.

Because the clavichord does not have a pluck or strike point, its spectrum does not possess the dips we saw for the guitar and piano. Interestingly, it still sounds somewhat like a guitar.

Future directions

Can you hear the shape of a vibrating string? For a guitar string, the answer is probably yes, because that case involves essentially only one variable, the plucking point's location. For the piano and clavichord, the answer is less clear because the string's initial state involves the displacement and velocity. You can only "hear" the P_n, and these might not contain enough information to uniquely determine the initial conditions for both the string shape (displacement) and velocity. A further complication concerns a piano string's initial motion while the hammer is still in contact (or a clavichord string before the tangent comes to rest). Intuitively, you might expect that the string's motion during the few milliseconds of contact time will leave some trace in the initial part of the tone. I suspect that reconstructing the string motion during this period on the basis of the tone will not be trivial.

If you are interested in exploring this topic further, see the sidebar.

Acknowledgments

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