

# Stability boundary analysis of nonlinear dynamics subject to state limits

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**Abstract:** In the spirit of Morse-Smale systems, the paper analyzes the structure of stability boundary of a stable equilibrium point for nonlinear dynamics subject to state limits. Presence of state limits implies that the underlying dynamics does not satisfy the Lipschitz condition for solution existence/uniqueness. There does not exist a smooth flow for the dynamics thus complicating traditional analysis of stability boundary. By analyzing geometric properties of the solutions of the constrained dynamics, the paper establishes a characterization of the stability boundary under rather strong assumptions as a first step towards detailed boundary characterization.

**Key words:** Nonlinear dynamics, state limits, state saturation, stability analysis.

## I. INTRODUCTION

Consider the class of nonlinear dynamics

$$\Sigma : \quad dx/dt = f(x), \quad x \in \mathbf{R}^n \quad (1)$$

where the states  $x$  are limited to

$$X = \{ x \in \mathbf{R}^n, x_{li} \leq x \leq x_{ui} \text{ for } i=1,2,\dots,n \} \quad (2)$$

The lower and upper limits  $x_{li}$  and  $x_{ui}$  are assumed to be  $-\infty$  and  $+\infty$ , if state  $x_i$  is not limited in the respective direction. We will assume that the function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is smooth in the neighborhood of the constrained state space  $X$ . Given the constrained dynamics (1) and (2), the first problem is to generate the modified “vector field” say  $\underline{f}(x)$  that renders the dynamics (1) consistent with the constrained formulation (2). In engineering sense, we say that the state  $x_i$  gets stuck at its limit say  $x_{ui}$  whenever the state is at  $x_{ui}$  and the state  $x_i$  is pushed above its limit by the derivative  $dx_i/dt=f_i(x) (>0)$ . The formulation of the modified vector field  $\underline{f}(x)$  can thus be summarized in an engineering sense as

$$\begin{aligned} \underline{f}_i(x) &= 0 && \text{if } x_i = x_{ui} \text{ and } f_i(x) \geq 0 \\ &= 0 && \text{if } x_i = x_{li} \text{ and } f_i(x) \leq 0 \\ &= f_i(x) && \text{otherwise} \end{aligned} \quad (3)$$

Hardlimits on state  $x_i$  are commonly known as “nonwindup limits” in the power engineering literature [1], while the name “state limits” is more common in the control literature and in circuits literature. Equation (3) implies that when a trajectory reaches a state limit surface say  $X_{ui} = \{ x_i = x_{ui} \}$ , and if the state  $x_i$  tries to move past the limit with  $f_i > 0$ , then the state  $x_i$  gets

stuck at the limit  $x_{ui}$ , and therefore,  $\underline{f}_i$  is annihilated to be zero. At these points then, the vector field  $\underline{f}$  will be discontinuous and specifically, is not locally Lipschitz. Therefore, it is not at all clear when solutions exist for the dynamics (1) and whether they are unique. This problem was analyzed in depth in [2] and we summarize the existence result first.

In order to construct the solutions of the dynamics (1)-(2) from the engineering definition (3), we need to make the following technical assumption [2]. Suppose  $x(t_0)=p$ , and say  $p_i = x_{ui}$  so that  $p$  belongs to the state limit set  $\{ x_i = x_{ui} \}$ . Then we say that the local solution say  $x(t)$  of (1)-(2) with  $x(t_0)=p$  has order of contact  $k$  with  $\{ x_i = x_{ui} \}$  at  $p$  if all the first  $k-1$  derivatives of the  $x_i$  coordinate of  $x(t)$  vanish at  $p$ , and the  $k$ -th derivative is nonzero.

**(SL0)** All non-constant solution curves of the dynamics (1)-(2) have finite order of contact with all the state limit surfaces, the boundary of  $X$ .

Details on assumption (SL0) and the subsequent mathematical construction of the solution sets for (1)-(2) can be seen in [2]. The following theorem from [2] summarizes the solution properties.

**Theorem 1 [2].** Under Assumption (SL0), there exists a unique vector field  $\underline{f}$  which is compatible with the state limits in the sense of dynamics (1)-(2). The vector field is piecewise smooth in the sense that there exists a decomposition of  $X$  into embedded submanifolds with the property that the restriction of  $\underline{f}$  onto these submanifolds are smooth vector fields. For every point  $p \in X$ , there exists a unique solution  $x(t)$  with  $x(0)=p$  forward in time that satisfies (1)-(2) with a maximal interval of existence  $[0,T(p))$ . The vector field  $\underline{f}$  generates a positive semi-flow.

For the state limited dynamics (1)-(2), unique solutions exist for any initial condition in the constrained state space  $X$  in forward time, whereas neither existence nor uniqueness holds for solutions in negative time.

We can now formulate the problem to be studied in this paper. Let us consider a stable equilibrium point  $x_s$  which belongs to the interior of  $X$ . Since solutions are well-defined in forward time, the region of attraction for  $x_s$  can be readily defined as

$$A = \{ p \in X : \text{the solution } x(t) \rightarrow x_s \text{ as } t \rightarrow \infty \\ \text{with } x(0)=p \} \quad (4)$$

For nonlinear dynamics such as in power system dynamics, the vital question is to determine whether a specified initial condition  $p_0 \in A$  or not. Lyapunov theoretic methods such as the controlling unstable equilibrium point method [1] in the power engineering literature, utilize the structure of the stability boundary  $\partial A$  of  $A$  for tackling this question.

For smooth systems, classical analysis by Morse and Smale (e.g. [3]) proves a full hierarchical structure of the stability boundary for dynamics restricted to compact manifolds under certain assumptions henceforth called Morse-Smale assumptions. In the engineering literature, the Morse-Smale characterization was extended to general smooth nonlinear systems in [4],[5]. A new concept of quasi-stability boundary which is defined as the boundary of the closure of the region of attraction  $\partial Cl(A)$  was proposed in [4]. The structure of the quasi-stability boundary  $\partial Cl(A)$  was shown in [4] to be simpler than that of the stability boundary  $\partial A$ . Using the concept of quasi-stability boundary and by using topological arguments, stability boundary characterization of singular differential-algebraic systems was established in [6],[7]. In this paper, we will analyze the quasi-stability boundary  $\partial Cl(A)$  for the state constrained nonlinear dynamics (1)-(2). Under certain Morse-Smale like assumptions, it is shown that the quasi-stability boundary  $\partial Cl(A)$  “mostly” consists of

- i) “stable manifolds” of certain unstable equilibrium points,
- ii) “stable manifolds” of certain unstable periodic orbits, and
- iii) some segments of the constraint boundary  $\partial X$ .

The characterization paves the way for extension of Lyapunov theory based direct stability analysis methods to nonlinear models containing state limits as well.

## II. STABILITY BOUNDARY ANALYSIS

Let us first define the state limit boundary sets  $X_{ui}$  and  $X_{li}$  as

$$X_{ui} = \{ x \in X : x_i = x_{ui}, \text{ and } x_j \neq x_{uj} \text{ and } x_j \neq x_{lj} \\ \text{for all } i \neq j \} \quad (5)$$

$$X_{li} = \{ x \in X : x_i = x_{li}, \text{ and } x_j \neq x_{uj} \text{ and } x_j \neq x_{lj} \\ \text{for all } i \neq j \} \quad (6)$$

From the engineering definition (3), for an initial condition  $x_0 \in X_{ui}$ , the local solution gets stuck on the surface  $X_{ui}$  if the derivative  $f_i(x_0) > 0$  so that the constrained vector field  $\underline{f}_i(x_0)$  becomes zero. On the other hand, if  $f_i(x_0) < 0$  for a point  $x_0 \in X_{ui}$ , then  $\underline{f}_i(x_0) = f_i(x_0) < 0$  and the local solution leaves the constraint set  $X_{ui}$  immediately. We call the first set of points in  $X_{ui}$  where  $f_i(x) > 0$  to be *active* and the second set of points with  $f_i(x) < 0$  to be *inactive* [2].

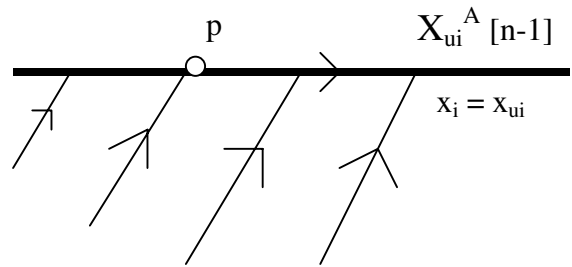


Fig. 1. Local phase portrait near a point  $p$  in  $X_{ui}^A$ .

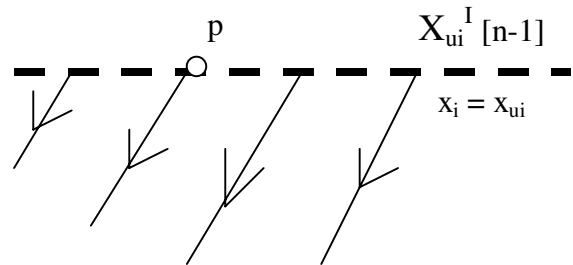


Fig. 2. Local phase portrait near a point  $p \in X_{ui}^I$ .

Motivated by this property, we can divide the set  $X_{ui}$  into  $X_{ui}^A$ ,  $X_{ui}^I$  and  $X_{ui}^0$  where  $f_i(x)$  is positive, negative and zero respectively. By (3), it follows that the active set  $X_{ui}^A$  is locally positive time-invariant since  $\underline{f}_i(x_0)=0$ . By similar arguments, we can divide the constraint set  $X_{li}$  into active subset  $X_{li}^A$ , inactive subset  $X_{li}^I$  and the rest  $X_{li}^0$ , depending on whether  $f_i(x)$  is negative, positive or zero respectively. Note that the vector field  $f(x)$  is transversal to both the active and inactive subsets  $X_{ui}^A$ ,  $X_{ui}^I$ ,  $X_{li}^A$ , and  $X_{li}^I$  while  $f(x)$  is tangential to the

sets  $X_{ui}^0$ , and  $X_{li}^0$ . The local phase portrait for any active point  $p \in X_{ui}^A$  is shown in Figure 1, while the local phase portrait at an inactive point  $p \in X_{ui}^I$  is summarized in Figure 2.

In Figures 1 and 2, the dimensions of the respective sets  $n-1$  are shown in brackets. In Figure 1, note that any solution in the vicinity of the active set  $X_{ui}^A$  from the interior of  $X$  reaches the set in finite time, and then the solution is locally stuck on the set  $X_{ui}^A$  in positive time. For any point  $p \in X_{ui}^A$ , there exists a unique solution in positive time that is locally invariant on  $X_{ui}^A$  while there exist infinitely many solutions in negative time. On the other hand, any solution starting from an inactive point such as  $p \in X_{ui}^I$  leaves the set immediately and enters the interior of  $X$  in positive time. Therefore, for a point  $p \in X_{ui}^I$ , there exists a unique solution locally in positive time that leaves the boundary of  $X$ , while there does not exist any solution in negative time. Similar analysis follows for sets  $X_{li}^I$  and  $X_{li}^I$  respectively.

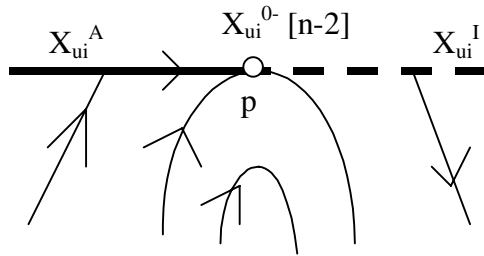


Fig. 3. Local phase portrait for a point  $p \in X_{ui}^{0-}$ .

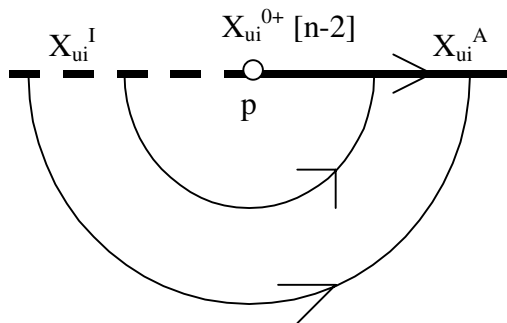


Fig. 4. Local phase portrait for a point  $p \in X_{ui}^{0+}$ .

Next, we need to consider the subsets  $X_{ui}^0$  and  $X_{li}^0$  respectively. Without loss of generality, we will consider the set  $X_{ui}^0$  only and similar analysis follows for  $X_{li}^0$ . At any point  $p \in X_{ui}^0$ , the first derivative  $dx_i/dt$

$= f_i(p) = 0$ , and therefore, we need to evaluate the second derivative  $d^2x/dt^2$  in order to determine the solution structure. Note that the second derivative  $d^2x/dt^2$  at  $p$  is given by the Lie derivative  $\partial f_i/\partial x f(x)$  at  $p$ . Therefore, depending on the sign of this term, the set  $X_{ui}^0$  can be further divided into  $X_{ui}^{0+}$ ,  $X_{ui}^{0-}$ , and  $X_{ui}^{00}$  respectively. The local phase portraits for any point  $p \in X_{ui}^{0+}$  and any point  $p \in X_{ui}^{0-}$  can be derived by straightforward analysis and are summarized in Figures 3 and 4 respectively. We can carry out the analysis further with the subsets  $X_{ui}^{00}$  and  $X_{li}^{00}$  by considering the next Lie derivatives. The exercise is deferred for a future paper. For stability boundary analysis, we will assume that these degenerate sets are empty.

(SL1) The sets  $X_{ui}^{00}$  and  $X_{li}^{00}$  are empty for all  $i=1,2,\dots,n$  in the closure of region of attraction  $cl(A)$ .

Thus far, we have analyzed the individual boundary segments  $X_{ui}$  and  $X_{li}$  where one of the state limits is reached and the other states are away from their limits. This follows directly from definitions (5) and (6). Technically, we can extend the analysis to the boundary segments between these sets namely, when two or more state limits are reached simultaneously (see [2] for details). In order to keep the analysis tractable, we will make a simplifying assumption in this paper that all trajectories of interest reach utmost one state limit at any time.

(SL2)  $\partial X \cap Cl(A) = (\cup_i (X_{ui} \cup X_{li})) \cap Cl(A)$ .

Next we need to develop some notions of stable and unstable manifolds for equilibrium points and periodic orbits for the constrained vector field  $\underline{f}(x)$ . For equilibrium points and periodic orbits which lie strictly in the interior of  $X$ , the definitions of local stable and unstable manifolds follow directly from nonlinear dynamical system theory. For equilibrium points which lie on the limit sets  $X_{ui}$  or  $X_{li}$ , their construction is nontrivial.

Let us recall from Theorem 1 that the vector field  $\underline{f}(x)$  induces a positive semi-flow on  $X$ . Therefore, the solutions are unique in forward time, and the definition of stable manifold is easy to state. For an equilibrium point  $x_0 \in X$  with  $\underline{f}(x_0)=0$ , we define the "stable manifold"  $W^s(x_0)$  as

$$W^s(x_0) = \{ x \in X : \text{the solution } x(t) \text{ with } x(0)=x \text{ satisfies } x(t) \rightarrow x_0 \text{ as } t \rightarrow \infty \} \quad (7)$$

For the state limited dynamics (1)-(2), there is an interesting possibility that a trajectory can reach an equilibrium  $x_0 \in \partial X$  in finite positive time when  $\underline{f}(x_0)=0$  and  $f(x_0) \neq 0$ . In this case also, the solution becomes constant subsequently, and hence the trajectory would belong to the stable manifold  $W^s(x_0)$  be definition (7). However, because of the presence of such nonsmooth trajectories in  $W^s(x_0)$ , the set  $W^s(x_0)$  will in general not be a manifold. The definition of “unstable manifolds” is even more tricky since the solutions are not unique and may not even exist in negative time. We will come back to unstable manifolds a little later in the paper.

Next, we can define a solution curve  $\gamma(t)$  with  $t \in [0, T]$  to be a periodic orbit for (1)-(2) if there exists a  $\tau > 0$  such that for any  $x_0 \in \gamma(t)$ , the solution with  $x(0)=x_0$  satisfies  $x(\tau)=x_0$ . For convenience, we will simply denote  $\gamma(t)$  for  $t \in [0, \tau]$  to be the periodic orbit  $\gamma$ . The definition of “stable manifold” for the periodic orbit  $\gamma$  follows directly from definition (7) by replacing  $x_0$  with  $\gamma$  in definition (7).

To impose a nicer geometric structure for the stable manifolds above, we next assume that the equilibrium points and periodic orbits do not intersect with any of the sets  $X_{ui}^0$  and  $X_{li}^0$ .

**(SL3)** Equilibrium points and periodic solutions of the vector field  $\underline{f}(x)$  do not intersect with any of  $X_{ui}^0$  and  $X_{li}^0$  for all  $i$ .

Then, it easily follows that an equilibrium say  $x_0$  if present on  $\partial X$  must be present in  $X_{ui}^A$  or  $X_{li}^A$  for some  $i$ . In that case, we know that the  $i$ -th component of vector field  $\underline{f}_i(x)=0$  locally near  $x_0$  on  $\partial X$ . That is, the vector field  $\underline{f}(x)$  is locally invariant on  $\partial X$ , and the remaining coordinates  $j \neq i$  form the easy choice locally for the chart coordinates of the restricted dynamics of  $\underline{f}(x)$  on  $\partial X$ . Next, we define an equilibrium point  $x_0$  in  $\partial X$  to be hyperbolic if  $x_0$  is hyperbolic for the restricted dynamics of  $\underline{f}$  on  $\partial X$ . Restricted to  $\partial X$ , the definitions of stable (say  $W_{loc}^s(x_0)$ ) and unstable (say  $W_{loc}^u(x_0)$ ) manifolds for hyperbolic equilibrium  $x_0$  are also then straightforward, and they are indeed smooth embedded submanifolds of  $\partial X$  with dimensions equal to the stable and unstable eigenspaces respectively, of the restricted dynamics. We define the local unstable manifold of  $x_0$  for the dynamics (1)-(2) to be simply that of the restricted dynamics of  $\underline{f}$  on  $\partial X$ . Therefore, the local unstable manifold is indeed a smooth manifold of dimension equal to the unstable eigenspace of  $\underline{f}$

restricted to  $\partial X$  at  $x_0$ . Next, since the solutions are defined uniquely in positive time, we can simply extend the local unstable manifold definitions into global definitions by extending the respective solutions over their maximum intervals of existence. However, we cannot guarantee the global extensions of the local unstable manifolds to be manifolds in a geometric or even topological sense. Loosely speaking however, we will denote the global extensions to be unstable manifolds in this paper.

Looking into the interior of  $X$ , the solutions nearby locally reach the active set  $X_{ui}^A$  or  $X_{li}^A$  in finite positive time as shown in Figure 1. Therefore, for every point  $x_1 \in W_{loc}^s(x_0)$ , there exists a unique local solution  $x(t) \subset \text{interior}(X)$  that reaches  $x_1$  in finite positive time. Therefore, it follows that the local stable manifold at  $x_0$  for the state constrained dynamics (1)-(2) is a  $(n_s+1)$ -dimensional manifold with boundary where  $n_s$  is the dimension of  $W_{loc}^s(x_0)$  restricted to  $\partial X$ . For the stable manifold of the constrained vector field  $\underline{f}$ , it follows that the boundary for the manifold is indeed  $W_{loc}^s(x_0)$  on  $\partial X$ .

We can construct the stable and unstable manifolds for periodic orbits in  $\partial X$  by similar arguments. In order to establish global properties of these sets, we make the following technical assumption.

**(SL4)** All equilibrium points and all periodic orbits in  $\text{cl}(A) \cap \text{int}(X)$  are hyperbolic. All equilibrium points and all periodic orbits in  $\text{cl}(A) \cap \partial X$  are hyperbolic for the respective restricted dynamics of  $\underline{f}$  on  $\partial X$ . The stable manifolds of all equilibrium points and all periodic orbits in  $\text{cl}(A)$  do not intersect with any of  $X_{ui}^0$  or  $X_{li}^0$  for all  $i$ .

With assumption (SL4), it can be shown that for any equilibrium point or periodic orbit in  $\text{cl}(A)$ , their stable manifolds are mostly contained in the interior of  $X$ , that is,  $\text{Cl}(W^s \cap \text{int}(X)) = W^s$ . This property allows us to develop a notion of transversal intersection of stable and unstable manifolds by restricting the definition to the interior of  $X$ .

We say that two sets  $W_1$  and  $W_2$  intersect transversally if the intersection of  $W_1 \cap \text{int}(X)$  and  $W_2 \cap \text{int}(X)$  is transversal. Equipped with this definition, we can now state Morse-Smale like assumptions for stability boundary analysis. Recall from Morse-Smale theory [4], [5] that the following assumptions (SL5) and (SL6)

are nongeneric. However, we can expect them to be satisfied in typical engineering models [4], [5].

**(SL5)** All intersections of stable manifolds of equilibrium points and periodic orbits with their unstable manifolds are transversal in the closure of  $A$ .

**(SL6)** For any initial condition  $x_0 \in \text{cl}(A)$ , the nonwandering set for the corresponding solution  $x(t)$  in positive time consists only of a finite number of equilibrium points and periodic orbits.

We can now present the main result of the paper on quasi-stability characterization. In the Theorem below, the superscripts for equilibrium points and periodic orbits denote the dimension of their stable manifolds in the interior of  $X$ .

**Theorem 2**

- (A)  $x^{n-1} \in \partial \text{cl}(A) \Leftrightarrow W^u(x^{n-1}) \cap A \neq \emptyset$  and  $W^u(x^{n-1}) \cap (\text{Cl}(A))^c \neq \emptyset$   
 (B)  $\gamma^{n-1} \subset \partial \text{cl}(A) \Leftrightarrow W^u(\gamma^{n-1}) \cap A \neq \emptyset$  and  $W^u(\gamma^{n-1}) \cap (\text{Cl}(A))^c \neq \emptyset$

Theorem 2 states that for equilibrium points and periodic orbits with  $(n-1)$ -dimensional stable manifolds in  $\partial \text{cl}(A)$ , at least one solution from the unstable manifold must converge to the stable equilibrium  $x_s$ , and at least one solution must diverge away.

Next, we can state the quasi-stability boundary composition to be as follows.

**Theorem 3**

$$\begin{aligned} \partial \text{cl}(A) = & \text{cl} \left( \left( \partial A \cap \bigcup_i (X_{ui}^1 \cup X_{ii}^1) \right) \right. \\ & \cup \left( \bigcup_{x^{n-1} \in \partial \text{cl}(A)} W^s(x^{n-1}) \right) \\ & \left. \cup \left( \bigcup_{\gamma^{n-1} \in \partial \text{cl}(A)} W^s(\gamma^{n-1}) \right) \right) \end{aligned}$$

The proofs of Theorem 2 and 3 are quite technical and will be presented elsewhere.

Theorem 3 can be summarized as follows. The quasi-stability boundary  $\partial \text{cl}(A)$  for the dynamics (1)-(2) consists of three principal components: 1) inactive boundary segments from  $X_{ui}^1$  and  $X_{ii}^1$ , 2)  $(n-1)$ -dimensional stable manifolds of equilibrium points in  $\partial \text{cl}(A)$ , and 3)  $(n-1)$ -dimensional stable manifolds of periodic orbits in  $\partial \text{cl}(A)$ .

An example of a state constrained dynamics is presented in Figure 5. While Figure 5 shows a planar example, the strength of the theory is in the fact that Theorems 2 and 3 are applicable for general  $n$ -dimensional systems. In Fig. 2,  $x_s$  denotes the stable equilibrium, and there is also a second stable equilibrium at the  $\partial X$  boundary point  $x_s' = (x_{11}, x_{u2})$ . The region of attraction for the stable equilibrium  $x_s$  is bounded by 1) subsets of inactive boundary segments  $X_{u2}^1$ ,  $X_{u1}^1$ , and  $X_{12}^1$ , and 2) stable manifold  $W^s(x_u)$  of the unstable equilibrium  $x_u$  on the boundary set  $X_{u2}^A$ . In this example, the stability boundary  $\partial A$  is also the quasi-stability boundary  $\partial \text{cl}(A)$ .

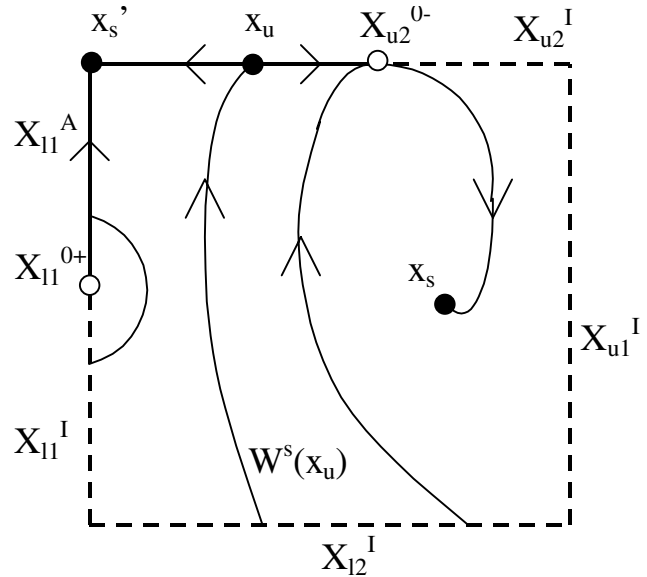


Fig. 5. A qualitative planar example

In summary, based on the results in this paper, it is indeed possible to extend Lyapunov theoretic methods for stability boundary analysis for state limited dynamics (1)-(2) satisfying Assumptions (SL0) – (SL6). Such methods will be presented elsewhere.

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